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# MATHEMATICS today

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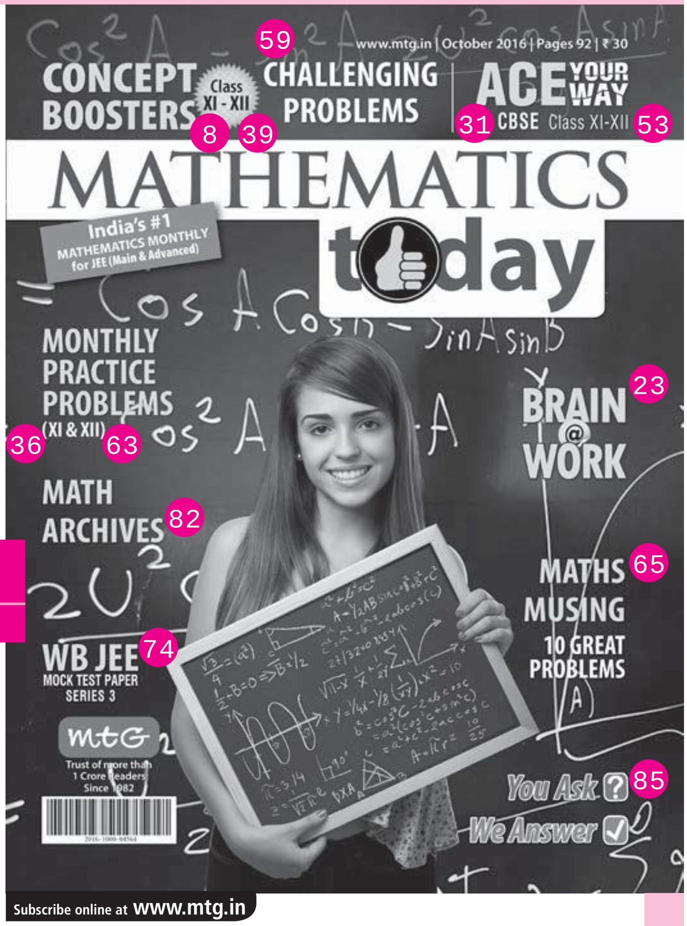


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Editor : Anil Ahlawat

# CONCEPT BOOSTERS

Class XI

## Straight Line

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

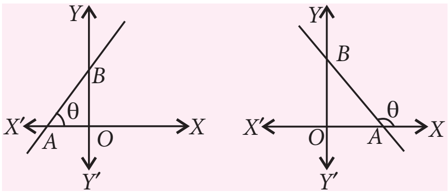
*ALOK KUMAR, B.Tech, IIT Kanpur

### DEFINITION

The straight line is a curve such that every point on the line segment joining any two points on it lies on it.

### SLOPE (GRADIENT) OF A LINE

The slope of a line is generally denoted by $m$. Thus, $m = \tan\theta$




- Slope of line parallel to $x$-axis is $m = \tan 0° = 0$
- Slope of line parallel to $y$-axis is $m = \tan 90° = \infty$
- Slope of the line equally inclined with the axes is 1 or –1.
- Slope of the line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\dfrac{y_2 - y_1}{x_2 - x_1}$ (taken in the same order).
- Slope of the line $ax + by + c = 0$, $b \neq 0$ is $-\dfrac{a}{b}$.
- Slope of two parallel lines are equal.
- If $m_1$ and $m_2$ be the slopes of two perpendicular lines, then $m_1 m_2 = -1$.

### EQUATION OF STRAIGHT LINE IN DIFFERENT FORMS

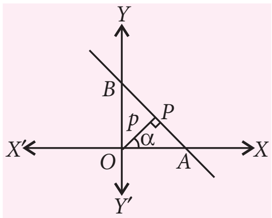
- **Slope form :** $y = mx$
- **Point slope form :** $y - y_1 = m(x - x_1)$
- **Slope intercept form :** $y = mx + c$
- **Intercept form :** $\dfrac{x}{a} + \dfrac{y}{b} = 1$
- **Two point form:** Equation of the line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

  $$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

  In the determinant form it is given as

  $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

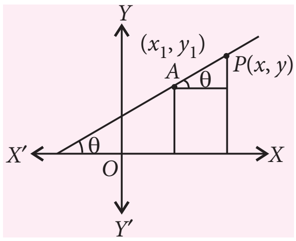
- **Normal or Perpendicular form :** $x\cos\alpha + y\sin\alpha = p$, where $p$ is the perpendicular distance of the line from the origin and $\alpha$ be the angle made by the perpendicular.




- **Symmetrical or Parametric or Distance form of a line :**

  $$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

  where $r$ is the distance between the point $P(x, y)$ and $A(x_1, y_1)$.

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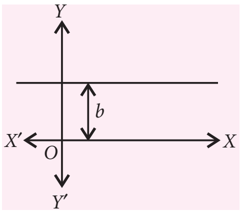
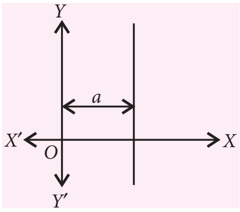
**Remarks**

- Equation of $x$-axis is $y = 0$.
- Equation of a line parallel to $x$-axis (or perpendicular to $y$-axis) at a distance of '$b$' from it is $y = b$.




- Equation of $y$-axis is $x = 0$.
- Equation of a line parallel to $y$-axis (or perpendicular to $x$-axis) at a distance of '$a$' from it is $x = a$




## EQUATION OF PARALLEL AND PERPENDICULAR LINES TO A GIVEN LINE

- Equation of a line which is parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$
- Equation of a line which is perpendicular to $ax + by + c = 0$ is $bx - ay + \lambda = 0$ where $\lambda$ is an arbitrary constant.

## GENERAL EQUATION OF A STRAIGHT LINE AND ITS TRANSFORMATION IN STANDARD FORMS

General equation of a line is $ax + by + c = 0$. It can be reduced in various standard forms given below.

- **Slope intercept form:** $y = -\frac{a}{b}x - \frac{c}{b}$, slope $m = -\frac{a}{b}$

  and intercept on $y$-axis is, $C = -\frac{c}{b}$

- **Intercept form :** $\frac{x}{-c/a} + \frac{y}{-c/b} = 1$, where $x$- intercept is $\left(-\frac{c}{a}\right)$ and $y$-intercept is $\left(-\frac{c}{b}\right)$

- **Normal form :** $-\frac{ax}{\sqrt{a^2+b^2}} - \frac{by}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}$

  where $\cos\alpha = -\frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = -\frac{b}{\sqrt{a^2+b^2}}$ and $p = \frac{c}{\sqrt{a^2+b^2}}$

## POINT OF INTERSECTION OF TWO LINES

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be two non-parallel lines. If $(x', y')$ be the co-ordinates of their point of intersection, then $a_1x' + b_1y' + c_1 = 0$ and $a_2x' + b_2y' + c_2 = 0$

Solving these equations, we get

$$(x', y') = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

$$= \left(\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}\right)$$

## GENERAL EQUATION OF LINES THROUGH THE INTERSECTION OF TWO GIVEN LINES

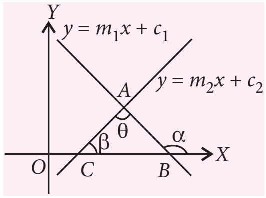
If equation of two lines $P : a_1x + b_1y + c_1 = 0$ and $Q : a_2x + b_2y + c_2 = 0$, then the equation of the line passing through the intersection of these lines is $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$.

## ANGLE BETWEEN TWO NON-PARALLEL LINES

- **When equations are in slope intercept form**

  Let $\theta$ be the angle between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$

  $\therefore \quad \theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$




- **When equations are in general form**

  The angle $\theta$ between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by $\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$.

## CONDITIONS FOR TWO LINES TO BE COINCIDENT, PARALLEL, PERPENDICULAR AND INTERSECTING

- Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are,

  (a) Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

  (b) Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

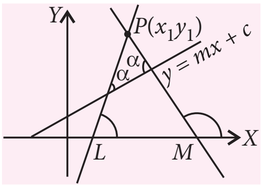
  (c) Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

  (d) Perpendicular, if $a_1a_2 + b_1b_2 = 0$

## EQUATION OF STRAIGHT LINES THROUGH A GIVEN POINT MAKING A GIVEN ANGLE WITH A GIVEN LINE

Let $P(x_1, y_1)$ be a given point and $y = mx + c$ be the given line. Let $\alpha$ be the angle made by that

line whose equation is to be determine with the given line $y = mx + c$ are




$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

## A LINE EQUALLY INCLINED WITH TWO LINES

Let the two lines with slopes $m_1$ and $m_2$ be equally inclined to a line with slope $m$

then $\frac{m_1 - m}{1 + m_1 m} = -\left(\frac{m_2 - m}{1 + m_2 m}\right)$

## BISECTORS OF THE ANGLES BETWEEN TWO STRAIGHT LINES

- **Bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$**

  (i) Containing the origin

  $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

  (ii) Not containing the origin

  $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{-(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

- **To find the acute and obtuse angle bisectors**

  Let θ be the angle between one of the lines and one of the bisectors given by

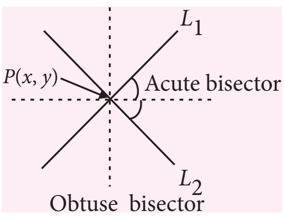
  $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

  Find tanθ. If $|\tan\theta| < 1$, then this bisector is the bisector of acute angle and the other one is the bisector of the obtuse angle.

  If $|\tan\theta| > 1$, then this bisector is the bisector of obtuse angle and other one is the bisector of the acute angle.

- **Method to find acute angle bisector and obtuse angle bisector**

  (i) If $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to "+" sign gives the obtuse angle bisector and the bisector corresponding to "–" sign is the bisector of acute angle between the lines.




  (ii) If $a_1a_2 + b_1b_2 < 0$, then the bisector corresponding to "+" and "–" sign given the acute and obtuse angle bisectors respectively.

**Remarks**

- Bisectors are perpendicular to each other.
- If $a_1a_2 + b_2b_2 > 0$, then the origin lies in obtuse angle and if $a_1a_2 + b_1b_2 < 0$, then the origin lies in acute angle.

## LENGTH OF PERPENDICULAR

- **Distance of a point from a line :** The length $p$ of the perpendicular from the point $(x_1, y_1)$ to the line $ax + by + c = 0$ is given by $p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

- **Distance between two parallel lines :** Let the two parallel lines be $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ then the distance between the lines is $d = \frac{\lambda}{\sqrt{(a^2 + b^2)}}$, where

  (i) $\lambda = |c_1 - c_2|$, if they are on the same side of origin.

  (ii) $\lambda = |c_1| + |c_2|$, if the origin $O$ lies between them.

## POSITION OF A POINT WITH RESPECT TO A LINE

Let the given line be $ax + by + c = 0$ and observing point is $(x_1, y_1)$, then

- If the same sign is found by putting $x = x_1$, $y = y_1$ and $x = 0$, $y = 0$ in equation of line, then the point $(x_1, y_1)$ is situated on the side of the origin.
- If the opposite sign is found by putting $x = x_1$, $y = y_1$ and $x = 0$, $y = 0$ in equation of line then the point $(x_1, y_1)$ is situated on the opposite side of the origin.

## CONCURRENT LINES

Three or more lines are said to be concurrent lines if they meet at a point.

- Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if,

  $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

- The condition for the lines $P = 0$, $Q = 0$ and $R = 0$ to be concurrent is that three constants $a, b, c$ (not all zero at the same time) can be obtained such that $aP + bQ + cR = 0$

## SOME IMPORTANT RESULTS

- Area of the triangle formed by the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$, is $\frac{1}{2}\left|\sum \frac{(c_1 - c_2)^2}{m_1 - m_2}\right|$.
- Area of the triangle made by the line $ax + by + c = 0$ with the co-ordinate axes is $\frac{c^2}{2|ab|}$.
- Area of the rhombus formed by the lines $ax \pm by \pm c = 0$ is $\left|\frac{2c^2}{ab}\right|$.
- Area of the parallelogram formed by the lines $a_1x + b_1y + c_1 = 0; a_2x + b_2y + c_2 = 0, a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$ is $\left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$.
- The foot of the perpendicular $(h, k)$ from $(x_1, y_1)$ to the line $ax + by + c = 0$ is given by $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$. Hence, the coordinates of the foot of perpendicular is $\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}, \frac{a^2y_1 - abx_1 - bc}{a^2 + b^2}\right)$.
- Area of parallelogram $A = \frac{p_1p_2}{\sin\theta}$, where $p_1$ and $p_2$ are the distances between parallel sides and $\theta$ is the angle between two adjacent sides.
- The equation of a line whose mid-point is $(x_1, y_1)$ in between the axes is $\frac{x}{x_1} + \frac{y}{y_1} = 2$.
- The equation of a straight line which makes a triangle with the axes of centroid $(x_1, y_1)$ is $\frac{x}{3x_1} + \frac{y}{3y_1} = 1$.

## PROBLEMS

### Single Correct Answer Type

**1.** A line $L$ is perpendicular to the line $5x - y = 1$ and the area of the triangle formed by the line $L$ and coordinate axes is 5. The equation of the line $L$ is

(a) $x + 5y = 5$ (b) $x + 5y = \pm 5\sqrt{2}$
(c) $x - 5y = 5$ (d) $x - 5y = 5\sqrt{2}$

**2.** If the coordinates of the points $A$, $B$, $C$ and $D$ be $(a, b)$, $(a', b')$, $(-a, b)$ and $(a', -b')$ respectively, then the equation of the line bisecting the line segments $AB$ and $CD$ is

(a) $2a'y - 2bx = ab - a'b'$
(b) $2ay - 2b'x = ab - a'b'$
(c) $2ay - 2b'x = a'b - ab'$
(d) none of these

**3.** If the middle points of the sides $BC$, $CA$ and $AB$ of the triangle $ABC$ be (1, 3), (5, 7) and (–5, 7) respectively, then the equation of the side $AB$ is

(a) $x - y - 2 = 0$ (b) $x - y + 12 = 0$
(c) $x + y - 12 = 0$ (d) none of these

**4.** The equation of the line perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$ and passing through the point at which it cuts $x$-axis, is

(a) $\frac{x}{a} + \frac{y}{b} + \frac{a}{b} = 0$ (b) $\frac{x}{b} + \frac{y}{a} = \frac{b}{a}$
(c) $\frac{x}{b} + \frac{y}{a} = 0$ (d) $\frac{x}{b} + \frac{y}{a} = \frac{a}{b}$

**5.** The equation of the line bisecting the line segment joining the points $(a, b)$ and $(a', b')$ at right angle, is

(a) $2(a - a')x + 2(b - b')y = a^2 + b^2 - a'^2 - b'^2$
(b) $(a - a')x + (b - b')y = a^2 + b^2 - a'^2 - b'^2$
(c) $2(a - a')x + 2(b - b')y = a'^2 + b'^2 - a^2 - b^2$
(d) none of these

**6.** The equation of the lines which passes through the point (3, –2) and are inclined at 60° to the line $\sqrt{3}x + y = 1$ are

(a) $y + 2 = 0,\ \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$
(b) $x - 2 = 0,\ \sqrt{3}x - y + 2 + 3\sqrt{3} = 0$
(c) $\sqrt{3}x - y - 2 - 3\sqrt{3} = 0$ (d) none of these

**7.** Equation of the line passing through (–1, 1) and perpendicular to the line $2x + 3y + 4 = 0$ is

(a) $2(y - 1) = 3(x + 1)$ (b) $3(y - 1) = -2(x + 1)$
(c) $y - 1 = 2(x + 1)$ (d) $3(y - 1) = x + 1$

**8.** The intercept cut off from $y$-axis is twice that from $x$-axis by the line and line is passes through (1, 2), then its equation is

(a) $2x + y = 4$ (b) $2x + y + 4 = 0$
(c) $2x - y = 4$ (d) $2x - y + 4 = 0$

**9.** The equation of line whose mid point is $(x_1, y_1)$ in between the axes, is

(a) $\frac{x}{x_1} + \frac{y}{y_1} = 2$ (b) $\frac{x}{x_1} + \frac{y}{y_1} = \frac{1}{2}$

(c) $\frac{x}{x_1}+\frac{y}{y_1}=1$ (d) none of these

**10.** The equation of line passing through the point of intersection of the lines $4x - 3y - 1 = 0$ and $5x - 2y - 3 = 0$ and parallel to the line $2y - 3x + 2 = 0$, is

(a) $x - 3y = 1$ (b) $3x - 2y = 1$
(c) $2x - 3y = 1$ (d) $2x - y = 1$

**11.** Equation of the hour hand at 4 O' clock is

(a) $x-\sqrt{3}\,y=0$ (b) $\sqrt{3}x-y=0$
(c) $x+\sqrt{3}y=0$ (d) $\sqrt{3}x+y=0$

**12.** Equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the $x$-axis, is

(a) $x\sqrt{3}+y+8=0$ (b) $x\sqrt{3}-y=8$
(c) $-x\sqrt{3}+y=8$ (d) $x-\sqrt{3}y+8=0$

**13.** The equations of two lines through $(0, a)$ which are at distance '$a$' units from the point $(2a, 2a)$ are

(a) $y - a = 0$ and $4x - 3y - 3a = 0$
(b) $y - a = 0$ and $3x - 4y + 3a = 0$
(c) $y - a = 0$ and $4x - 3y + 3a = 0$
(d) none of these

**14.** A line is such that its segment between the straight lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$, then its equation is

(a) $83x - 35y + 92 = 0$
(b) $35x - 83y + 92 = 0$
(c) $35x + 35y + 92 = 0$
(d) none of these

**15.** The equations of the lines through the point of intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point $(3, 2)$ is $7/5$ is

(a) $3x - 4y - 6 = 0$ and $4x + 3y + 1 = 0$
(b) $3x - 4y + 6 = 0$ and $4x - 3y - 1 = 0$
(c) $3x - 4y + 6 = 0$ and $4x - 3y + 1 = 0$
(d) none of these

**16.** The number of lines that are parallel to $2x + 6y + 7 = 0$ and have an intercept of length 10 between the co-ordinate axes is

(a) 1 (b) 2
(c) 4 (d) infinitely many

**17.** The point $P(a, b)$ lies on the straight line $3x + 2y = 13$ and the point $Q(b, a)$ lies on the straight line $4x - y = 5$, then the equation of line $PQ$ is

(a) $x - y = 5$ (b) $x + y = 5$
(c) $x + y = -5$ (d) $x - y = -5$

**18.** Equation of a line passing through the point of intersection of lines $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and perpendicular to $6x - 7y + 3 = 0$, then its equation is

(a) $119x + 102y + 125 = 0$
(b) $119x + 102y = 125$
(c) $119x - 102y = 125$
(d) none of these

**19.** The equation of the line bisecting perpendicularly the segment joining the points $(-4, 6)$ and $(8, 8)$ is

(a) $6x + y - 19 = 0$ (b) $y = 7$
(c) $6x + 2y - 19 = 0$ (d) $x + 2y - 7 = 0$

**20.** The opposite angular points of a square are $(3, 4)$ and $(1, -1)$. Then the co-ordinates of other two points are

(a) $D\left(\frac{1}{2},\frac{9}{2}\right), B\left(-\frac{1}{2},\frac{5}{2}\right)$
(b) $D\left(\frac{1}{2},\frac{9}{2}\right), B\left(\frac{1}{2},\frac{5}{2}\right)$
(c) $D\left(\frac{9}{2},\frac{1}{2}\right), B\left(-\frac{1}{2},\frac{5}{2}\right)$
(d) none of these

**21.** Two consecutive sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation to one diagonal is $11x + 7y = 9$, then the equation of the other diagonal is

(a) $x + 2y = 0$ (b) $2x + y = 0$
(c) $x - y = 0$ (d) none of these

**22.** The equation of the lines on which the perpendiculars from the origin make 30° angle with $x$-axis and which form a triangle of area $\frac{50}{\sqrt{3}}$ with axes, are

(a) $x+\sqrt{3}y\pm 10=0$ (b) $\sqrt{3}x+y\pm 10=0$
(c) $x\pm\sqrt{3}y-10=0$ (d) none of these

**23.** The base $BC$ of a triangle $ABC$ is bisected at the point $(p, q)$ and the equations to the sides $AB$ and $AC$ are respectively $px + qy = 1$ and $qx + py = 1$. Then the equation of the median through $A$ is

(a) $(2pq-1)(px+qy-1)=(p^2+q^2-1)(qx+py-1)$
(b) $(p^2 + q^2 - 1)(px + qy - 1) = (2p - 1)(qx + py - 1)$
(c) $(pq-1)(px+qy-1)=(p^2+q^2-1)(qx+py-1)$
(d) none of these

**24.** A straight line moves so that the sum of the reciprocals of its intercepts on two perpendicular lines is constant, then the line passes through

(a) a fixed point (b) a variable point
(c) origin (d) none of these

**25.** If $u \equiv a_1x + b_1y + c_1 = 0$ and $v \equiv a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the curve $u + kv = 0$ is
(a) the same straight line $u$
(b) different straight line
(c) it is not a straight line
(d) none of these

**26.** If $a + b + c = 0$ and $p \neq 0$, the lines $ax + (b + c)y = p$, $bx + (c + a)y = p$ and $cx + (a + b)y = p$
(a) do not intersect (b) intersect
(c) are concurrent (d) none of these

**27.** The equation of the straight line passing through the point (4, 3) and making intercepts on the co-ordinate axes whose sum is – 1, is
(a) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$
(b) $\frac{x}{2}-\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$
(c) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{2}+\frac{y}{1}=1$
(d) $\frac{x}{2}+\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$

**28.** The line which is parallel to $x$-axis and crosses the curve $y=\sqrt{x}$ at an angle of 45° is equal to
(a) $x = 1/4$ (b) $y = 1/4$
(c) $y = 1/2$ (d) $y = 1$

**29.** The line parallel to the $x$-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$ is
(a) above the $x$-axis at a distance of 3/2 from it
(b) above the $x$-axis at a distance of 2/3 from it
(c) below the $x$-axis at a distance of 3/2 from it
(d) below the $x$-axis at a distance of 2/3 from it

**30.** The equation to the line bisecting the join of (3, –4) and (5, 2) and having its intercepts on the $x$-axis and the $y$-axis in the ratio 2 : 1 is
(a) $x + y - 3 = 0$ (b) $2x - y = 9$
(c) $x + 2y = 2$ (d) $2x + y = 7$

**31.** The line $lx + my + n = 0$ will be parallel to $x$-axis, if
(a) $l = m = 0$ (b) $m = n = 0$
(c) $l = n = 0$ (d) $l = 0$

**32.** To which of the following types the straight lines represented by $2x + 3y - 7 = 0$ and $2x + 3y - 5 = 0$ belong?
(a) parallel to each other
(b) perpendicular to each other
(c) inclined at 45° to each other
(d) coincident pair of straight lines

**33.** The line passes through (1, 0) and $(-2, \sqrt{3})$ makes an angle of ...... with $x$-axis.
(a) 60° (b) 120° (c) 150° (d) 135°

**34.** Angle between $x = 2$ and $x - 3y = 6$ is
(a) $\infty$ (b) $\tan^{-1}(3)$
(c) $\tan^{-1}(1/3)$ (d) none of these

**35.** A straight line through origin bisect the line passing through the given points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$, then the lines are
(a) perpendicular
(b) parallel
(c) angle between them is $\pi/4$
(d) none of these

**36.** In $\Delta ABC$, $P$ is any point inside a triangle such that area of $\Delta BPC$, $\Delta APC$, $\Delta APB$ are equal. Line $AP$ cut $BC$ at $M$, area of $\Delta PMC$ is 5 sq. units, then area of $\Delta ABC$ is
(a) 20 sq. units (b) 25 sq. units
(c) 30 sq. units (d) 10 sq. units

## Multiple Correct Answer Type

**37.** Let $u \equiv ax + by + a\sqrt[3]{b} = 0$, $v \equiv bx - ay + b\sqrt[3]{a} = 0$, $a, b \in R$ be two straight lines. The equation of the bisectors of the angle formed by $L_1 \equiv (\tan\theta_1)u - (\tan\theta_2)v = 0$ and $L_2 \equiv (\tan\theta_1)u + (\tan\theta_2)v = 0$ for $\theta_1, \theta_2 \in \left(0, \frac{\pi}{2}\right)$ is
(a) $u = 0$
(b) $(\tan\theta_2)u + (\tan\theta_1)v = 0$
(c) $(\tan\theta_2)u - (\tan\theta_1)v = 0$
(d) $v = 0$

**38.** Equations of bisectors of angles between intersecting lines $\frac{x-3}{\cos\theta} = \frac{y+5}{\sin\theta}$, $\frac{x-3}{\cos\phi} = \frac{y+5}{\sin\phi}$ are $\frac{x-3}{\cos\alpha} = \frac{y+5}{\sin\alpha}$ and $\frac{x-3}{\beta} = \frac{y+5}{\gamma}$ then
(a) $\alpha = \frac{\theta+\phi}{2}$ (b) $\beta^2 + \gamma^2 = 1$
(c) $\tan\alpha = \frac{-\beta}{\gamma}$ (d) $\tan\alpha = \frac{\beta}{\gamma}$

**39.** If (3, 2), (–4, 1) and (–5, 8) are vertices of triangle then

(a) orthocentre is (4, 1)
(b) orthocentre is (–4, 1)
(c) circumcentre is (–1, 5)
(d) circumcentre is (3, 2)

**40.** The point $A$ divides the join of $P$ (–5, 1) and $Q$ (3, 5) in the ratio $k$ : 1. The values of $k$ for which the area of $\Delta ABC$ where $B$ (1, 5), $C$ (7, –2) is equal to 2 sq. units are

(a) 7 (b) 4 (c) $\frac{30}{4}$ (d) $\frac{31}{9}$

**41.** If the straight line $3x + 4y = 24$ intersect the axes at $A$ and $B$ and the straight line $4x + 3y = 24$ intersect the axes at $C$ and $D$ then points $A, B, C, D$ lie on

(a) the circle (b) the parabola
(c) an ellipse (d) the hyperbola

**42.** If $6a^2 - 3b^2 - c^2 + 7ab - ac + 4bc = 0$ then the family of lines $ax + by + c = 0$, $|a| + |b| \neq 0$ is concurrent at

(a) (–2, –3) (b) (3, –1)
(c) (2, 3) (d) (–3, 1)

## Comprehension Type

### Paragraph for Q. No. 43 to 45

$ABCD$ is a parallelogram whose side lengths are $a$ and $b$ $(a \neq b)$. The angular bisectors of interior angles are drawn to intersect one another to form quadrilateral. Let '$\alpha$' be one angle of parallelogram.

**43.** The area of the quadrilateral formed by the angular bisectors is

(a) $\frac{1}{2}(a-b)^2 \sin\frac{\alpha}{2}$ (b) $\frac{1}{2}(a-b)^2 \sin\alpha$

(c) $\frac{1}{2}(a-b)^2 \cos\frac{\alpha}{2}$ (d) $\frac{1}{2}(a-b)^2 \cos\alpha$

**44.** If $S$ is the area of the given parallelogram and $Q$ is the area of the quadrilateral formed by the angular bisectors then ratio of the larger side to smaller side of the parallelogram is

(a) $\frac{(S+Q)}{S}$ (b) $\frac{S+Q+\sqrt{2QS}}{S}$

(c) $\frac{S+Q+\sqrt{Q^2+2QS}}{S}$ (d) $\frac{S+Q+\sqrt{Q^2-2QS}}{S}$

**45.** The sides of the quadrilateral formed by the angular bisectors where $(a > b)$

(a) $(a-b)\sin\frac{\alpha}{2}$, $(a-b)\cos\frac{\alpha}{2}$

(b) $(a+b)\sin\frac{\alpha}{2}$, $(a+b)\cos\frac{\alpha}{2}$

(c) $(a - b)\sin\alpha$, $(a - b)\cos\alpha$

(d) $(a + b)\sin\alpha$, $(a + b)\cos\alpha$

## Matrix-Match Type

**46.** Match the following.

| | Column-I | | Column-II |
|---|---|---|---|
| (A) | The area bounded by the curve max $\{|x|, |y|\} = 1/2$ (in sq. units) is | (p) | 0 |
| (B) | If the point $(a, a)$ lies between the lines $|x + y| = 6$, then $[|a|]$ (where $[\cdot]$ denotes the greatest integer function) is | (q) | 1 |
| (C) | Number of non-zero integral values of $b$ for which the origin and the point (1, 1) lies on the same side of the straight line $a^2x + aby + 1 = 0$ for all $a \in R \sim \{0\}$ is | (r) | 2 |
| | | (s) | –2 |

## Integer Answer Type

**47.** A point $P(x, y)$ moves in such a way that $[x + y + 1] = [x]$ (where $[\cdot]$ denotes greatest integer function) and $x \in (0, 2)$. Then the area representing all the possible positions of $P$ equals

**48.** Given a point (2, 1). If the minimum perimeter of a triangle with one vertex at (2, 1), one on the $x$-axis, and one on the line $y = x$, is $k$, then $[k]$ is equal to (where $[\cdot]$ denotes the greatest integer function)

**49.** $ABCD$ and $PQRS$ are two variable rectangles, such that $A, B, C$ and $D$ lie on $PQ, QR, RS$ and $SP$ respectively and perimeter '$x$' of $ABCD$ is constant. If the maximum area of $PQRS$ is 32, then $x/4$ =

**50.** The area of the triangular region in the first quadrant, bounded above by the line $7x + 4y = 168$ and bounded below by the line $5x + 3y = 121$ is $\frac{7}{K}$, then the sum of digits of $K$ is

## SOLUTIONS

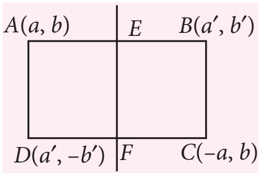
**1. (b) :** A line perpendicular to the line $5x - y = 1$ is given by $x + 5y - \lambda = 0 = L$, (given)

In intercept form : $\frac{x}{\lambda} + \frac{y}{\lambda/5} = 1$

So, area of triangle $= \frac{1}{2}(\lambda) \times \left(\frac{\lambda}{5}\right) = 5 \Rightarrow \lambda = \pm 5\sqrt{2}$

Hence, the equation of required straight lines is $x + 5y = \pm 5\sqrt{2}$

**2. (b) :** Mid point of




$AB \equiv E\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)$

and mid point of

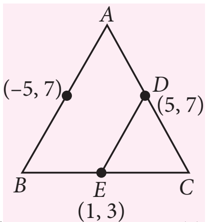
$CD \equiv F\left(\frac{a'-a}{2}, \frac{b-b'}{2}\right)$.

Hence equation of line $EF$ is

$$y - \left(\frac{b+b'}{2}\right) = \frac{b-b'-b-b'}{a'-a-a-a'}\left(x - \frac{a+a'}{2}\right)$$

On simplification, we get $2ay - 2b'x = ab - a'b'$

**3. (b) :** Slope of $DE = \frac{7-3}{5-1} = 1$




$\Rightarrow$ Slope of $AB = 1$

Hence equation of $AB$ is

$y - 7 = (x + 5)$

$\Rightarrow x - y + 12 = 0$

**4. (d) :** The given line is $bx - ay = ab$ ...(i)

Obviously it cuts $x$-axis at $(a, 0)$. The equation of line perpendicular to (i) is $ax + by = k$, but it passes through $(a, 0) \Rightarrow k = a^2$.

Hence required equation of line is $ax + by = a^2$

i.e. $\frac{x}{b} + \frac{y}{a} = \frac{a}{b}$

**5. (a) :** Slope $(m) = \frac{-1}{\frac{b'-b}{a'-a}} = \frac{a'-a}{b-b'}$.

Mid point of the given points is $\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)$

Therefore required equation of line is

$$y - \left(\frac{b+b'}{2}\right) = \frac{a'-a}{b-b'}\left(x - \frac{a+a'}{2}\right)$$

$\Rightarrow 2(b - b')y + 2(a - a')x - b^2 + b'^2 - a^2 + a'^2 = 0$

**6. (a) :** The equation of any straight line passing through $(3, -2)$ is $y + 2 = m(x - 3)$ ...(i)

The slope of the given line is $-\sqrt{3}$.

So, $\tan 60^o = \pm\frac{m - (-\sqrt{3})}{1 + m(-\sqrt{3})}$

On solving, we get $m = 0$ or $\sqrt{3}$

Putting the values of $m$ in (i), the required equation of lines are $y + 2 = 0$ and $\sqrt{3}x - y = 2 + 3\sqrt{3}$.

**7. (a) :** The gradient of line $2x + 3y + 4 = 0$ is $-2/3$. Now the equation of line passing through $(-1, 1)$ and having slope $m = -\frac{1}{-2/3} = \frac{3}{2}$ is $2(y - 1) = 3(x + 1)$.

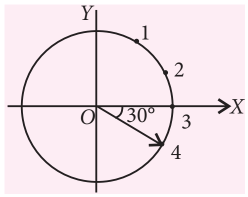
**8. (a) :** Let the intercepts be $a$ and $2a$, then the equation of line is $\frac{x}{a} + \frac{y}{2a} = 1$, but it also passes through $(1, 2)$, therefore $a = 2$.

Hence the required equation is $2x + y = 4$.

**9. (a) :** Intersection point on $x$-axis is $(2x_1, 0)$ and on $y$-axis is $(0, 2y_1)$. Thus equation of line passes through these points is $\frac{x}{x_1} + \frac{y}{y_1} = 2$

**10. (b) :** The point of intersection of the lines $4x - 3y - 1 = 0$ and $5x - 2y - 3 = 0$ is $(1, 1)$. The equation of line parallel to $2y - 3x + 2 = 0$ is $2y - 3x + k = 0$. It also passes through $(1, 1)$, therefore $k = 1$. Hence the required equation is $2y - 3x + 1 = 0$ or $3x - 2y = 1$

**11. (c) :** Since the hour, minute and second hands always pass through origin because one end of these hands is always at origin. Now at 4 O' clock, the hour hand makes 30° angle in fourth quadrant.




So the equation of hour hand is

$$y = mx \Rightarrow y = -\frac{1}{\sqrt{3}}x \Rightarrow x + \sqrt{3}y = 0$$

**12. (a) :** Slope $= -\sqrt{3}$

$\therefore$ Equation of line is $y = -\sqrt{3}x + c \Rightarrow \sqrt{3}x + y = c$

Now $\frac{c}{2} = |4| \Rightarrow c = \pm 8 \Rightarrow x\sqrt{3} + y = \pm 8$

**13. (c) :** Equation of any line through $(0, a)$ is

$y - a = m(x - 0)$ or $mx - y + a = 0$ ... (i)

If the length of perpendicular from $(2a, 2a)$ to the line (i) is '$a$', then $a = \pm\frac{m(2a) - 2a + a}{\sqrt{m^2+1}} \Rightarrow m = 0, \frac{4}{3}$.

Hence the required equation of lines are

$y - a = 0$, $4x - 3y + 3a = 0$

**14. (a) :** Any line through the middle point $M(1, 5)$ of the intercept $AB$ may be taken as

$$\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=r \quad \text{...(i)}$$

where '$r$' is the distance of any point $(x, y)$ on the line (i) from the point $M(1, 5)$.

Since the points $A$ and $B$ are equidistant from $M$ and on the opposite sides of it, therefore if the co-ordinates of $A$ are obtained by putting $r = d$ in (i), then the co-ordinates of $B$ are given by putting $r = -d$.

Now the point $A(1 + d\cos\theta, 5 + d\sin\theta)$ lies on the line $5x - y - 4 = 0$ and point $B(1 - d\cos\theta, 5 - d\sin\theta)$ lies on the line $3x + 4y - 4 = 0$.

Therefore, $5(1 + d\cos\theta) - (5 + d\sin\theta) - 4 = 0$
and $3(1 - d\cos\theta) + 4(5 - d\sin\theta) - 4 = 0$

Eliminating '$d$', we get $\dfrac{\cos\theta}{35}=\dfrac{\sin\theta}{83}$.

Hence the required equation of line is

$$\frac{x-1}{35}=\frac{y-5}{83} \quad \text{or} \quad 83x - 35y + 92 = 0$$

**15. (c) :** Point of intersection is $(2, 3)$. Therefore, the equation of line passing through $(2, 3)$ is

$$y - 3 = m(x - 2) \text{ or } mx - y - (2m - 3) = 0 \quad \text{...(i)}$$

According to the condition,

$$\frac{3m-2-(2m-3)}{\sqrt{1+m^2}}=\pm\frac{7}{5}\Rightarrow m=\frac{3}{4},\frac{4}{3}$$

Hence the equations are

$$3x - 4y + 6 = 0 \text{ and } 4x - 3y + 1 = 0.$$

**16. (b) :** The equation of any line parallel to $2x + 6y + 7 = 0$ is $2x + 6y + k = 0$.

This meets the axes at $A\left(-\dfrac{k}{2},0\right)$ and $B\left(0,-\dfrac{k}{6}\right)$.

By hypothesis, $AB = 10$

$$\Rightarrow \sqrt{\frac{k^2}{4}+\frac{k^2}{36}}=10\Rightarrow\sqrt{\frac{10k^2}{36}}=10$$

$$\Rightarrow 10k^2=3600 \Rightarrow k=\pm6\sqrt{10}$$

Hence there are two lines given by

$$2x+6y\pm6\sqrt{10}=0$$

**17. (b) :** Point $P(a, b)$ is on $3x + 2y = 13$

$\Rightarrow \quad 3a + 2b = 13 \quad$ ...(i)

Point $Q(b, a)$ is on $4x - y = 5$

$\Rightarrow \quad 4b - a = 5 \quad$ ...(ii)

By solving (i) and (ii), we get, $a = 3$, $b = 2$

$P(a, b) \equiv (3, 2)$ and $Q(b, a) \equiv (2, 3)$

Now, equation of $PQ$ is

$$y-2=\frac{3-2}{2-3}(x-3)$$

$\Rightarrow \quad y - 2 = -(x - 3) \quad \Rightarrow \quad x + y = 5$

**18. (b) :** The point of intersection of lines $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ is $\left(\dfrac{-2}{34},\dfrac{22}{17}\right)$

The slope of required line $=\dfrac{-7}{6}$

$\therefore$ Equation of required line is

$$y-\frac{22}{17}=\frac{-7}{6}\left(x+\frac{2}{34}\right) \Rightarrow 119x + 102y = 125$$

**19. (a) :** Equation of the line passing through $(-4, 6)$ and $(8, 8)$ is

$$y-6=\left(\frac{8-6}{8+4}\right)(x+4) \Rightarrow y-6=\frac{2}{12}(x+4)$$

$\Rightarrow \quad 6y - 36 = x + 4 \quad \Rightarrow \quad 6y - x - 40 = 0 \quad$ ... (i)

Now equation of any line perpendicular to (i) is

$6x + y + \lambda = 0 \quad$ ... (ii)

This line passes through the mid point of $(-4, 6)$ and $(8, 8)$ *i.e.*, $(2, 7)$

$\therefore \quad 6 \times 2 + 7 + \lambda = 0$

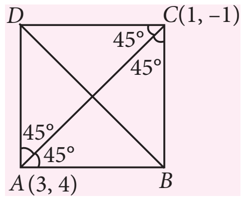
$\Rightarrow \quad 19 + \lambda = 0 \Rightarrow \lambda = -19$

From (ii), the required equation of line is

$6x + y - 19 = 0$

**20. (c) :** Obviously, slope of $AC = 5/2$.

Let $m$ be the slope of a line inclined at an angle of 45° to $AC$, then




$$\tan 45°=\pm\frac{m-\frac{5}{2}}{1+\frac{5}{2}m} \Rightarrow m=-\frac{7}{3},\frac{3}{7}$$

Let the slope of $AB$ or $DC$ be 3/7and that of $AD$ or $BC$ be $-7/3$. Then equation of $AB$ is $3x - 7y + 19 = 0$.

Also the equation of $BC$ is $7x + 3y - 4 = 0$.

On solving these equations, we get $B\equiv\left(-\dfrac{1}{2},\dfrac{5}{2}\right)$.

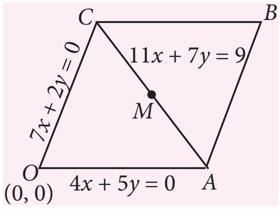
Now let the coordinates of the vertex $D$ be $(h, k)$. Since the middle points of $AC$ and $BD$ are same, therefore

$$\frac{1}{2}\left(h-\frac{1}{2}\right)=\frac{1}{2}(3+1)\Rightarrow h=\frac{9}{2}$$

And, $\dfrac{1}{2}\left(k+\dfrac{5}{2}\right)=\dfrac{1}{2}(4-1) \Rightarrow k=\dfrac{1}{2}$

Hence, $D\equiv\left(\dfrac{9}{2},\dfrac{1}{2}\right)$.

**21. (c) :** Since equation of diagonal $11x + 7y = 9$ does not pass through origin, so it cannot be the equation of the diagonal $OB$. Thus on solving the equation $AC$ with the equations $OA$ and $OC$, we get




$A\left(\frac{5}{3}, -\frac{4}{3}\right)$ and $C\left(\frac{-2}{3}, \frac{7}{3}\right)$

Therefore, the middle point $M$ is $\left(\frac{1}{2}, \frac{1}{2}\right)$

Hence the equation of $OB$ is $y = x$ *i.e.*, $x - y = 0$.

**22. (b) :** Let $p$ be the length of the perpendicular from the origin on the required line. Then its equation in normal form is $x\cos30° + y\sin30° = p$ or $\sqrt{3}x + y = 2p$

This meets the coordinate axes at $A\left(\frac{2p}{\sqrt{3}}, 0\right)$ and $B(0, 2p)$

$\therefore$ Area of $\Delta OAB = \frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)2p = \frac{2p^2}{\sqrt{3}}$

By hypothesis, $\frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \Rightarrow p = \pm 5$

$\therefore$ The required equation of line is $\sqrt{3}x + y = \pm 10$

**23. (a) :** The median passes through $A$, *i.e.*, the intersection of the given lines. Its equation is given by $(px + qy - 1) + \lambda(qx + py - 1) = 0$, where $\lambda$ is some real number. Also, since the median passes through the point $(p, q)$, we have $(p^2 + q^2 - 1) + \lambda(qp + pq - 1) = 0$.

$\Rightarrow \lambda = -\frac{p^2+q^2-1}{2pq-1}$ and the equation of median through $A$ is $(px+qy-1) - \frac{p^2+q^2-1}{2pq-1}(qx+py-1) = 0$

$\Rightarrow (2pq-1)(px+qy-1) = (p^2+q^2-1)(qx+py-1)$

**24. (a) :** Take two perpendicular lines as the coordinate axes. If $a$, $b$ be the intercepts made by the moving line on the coordinate axes, then the equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{....(i)}$$

According to the question, $\frac{1}{a} + \frac{1}{b} = \frac{1}{k}$ (say)

*i.e.*, $\frac{k}{a} + \frac{k}{b} = 1$ ....(ii)

The result (ii) shows that the straight line (i) passes through a fixed point $(k, k)$.

**25. (a) :** $u \equiv a_1x + b_1y + c_1 = 0$, $v \equiv a_2x + b_2y + c_2 = 0$

and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = c$ (say)

$\Rightarrow a_2 = \frac{a_1}{c}, b_2 = \frac{b_1}{c}, c_2 = \frac{c_1}{c}$

Given that $u + kv = 0$

$\Rightarrow a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$

$\Rightarrow a_1x + b_1y + c_1 + k\frac{a_1}{c}x + k\frac{b_1}{c}y + k\frac{c_1}{c} = 0$

$\Rightarrow a_1x\left(1+\frac{k}{c}\right) + b_1y\left(1+\frac{k}{c}\right) + c_1\left(1+\frac{k}{c}\right) = 0$

$\Rightarrow a_1x + b_1y + c_1 = 0$

$\Rightarrow u = 0$

**26. (a) :** By the help of given condition of $a + b + c = 0$, the three lines reduce to $x - y = \frac{p}{a}$ or $\frac{p}{b}$ or $\frac{p}{c}$ $(p \neq 0)$.

All these lines are parallel. Hence they do not intersect in finite plane.

**27. (a) :** Here $a + b = -1$.

Required line is $\frac{x}{a} - \frac{y}{1+a} = 1$ ....(i)

Since line (i) passes through (4, 3)

$\therefore \frac{4}{a} - \frac{3}{1+a} = 1 \Rightarrow 4 + 4a - 3a = a + a^2$

$\Rightarrow a^2 = 4 \Rightarrow a = \pm 2$

$\therefore$ Required lines are $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$

**28. (c) :** Let the equation of line parallel to $x$-axis be $y = \lambda$ ....(i)

Solving (i) with the cuve $y = \sqrt{x}$ ....(ii)

We get $P(\lambda^2, \lambda)$ which is the point of intersection at $P$

$\therefore$ Slope of (ii) is $m = \left(\frac{dy}{dx}\right)_{\text{at } P} = \frac{1}{2\lambda}$

$\because$ (i) and (ii) intersect at 45°

$\therefore \tan^{-1}\left(\frac{m-0}{1+m\cdot 0}\right) = \pm 45°$

$\Rightarrow m = \left(\frac{1}{2\lambda}\right) = \pm 1 \Rightarrow \lambda = \pm\frac{1}{2}$

$\therefore$ The equation of line is $y = \frac{1}{2}$ or $y = \frac{-1}{2}$

**29. (c) :** The lines passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is

$ax + 2by + 3b + \lambda(bx - 2ay - 3a) = 0$

$\Rightarrow (a + b\lambda)x + (2b - 2a\lambda)y + 3b - 3\lambda a = 0$ ...(i)

Line (i) is parallel to $x$-axis,

$\therefore\ a+b\lambda=0 \Rightarrow \lambda=\frac{-a}{b}$

Put the value of $\lambda$ in (i), we get

$$y\left(2b+\frac{2a^2}{b}\right)+3b+\frac{3a^2}{b}=0$$

$$\Rightarrow\ y\left(\frac{2b^2+2a^2}{b}\right)=-\left(\frac{3b^2+3a^2}{b}\right)$$

$$\Rightarrow\ y=\frac{-3}{2}$$

So, it is 3/2 units below $x$-axis.

**30. (c) :** Given equation of line having its intercepts on the $x$-axis and $y$-axis in the ratio 2 : 1 *i.e.*, $2a$ and $a$

$\therefore\ \frac{x}{2a}+\frac{y}{a}=1 \Rightarrow x+2y=2a$ ....(i)

According to question,

Line (i) also passes through midpoint of (3, –4) and (5, 2) *i.e.*, (4, –1).

$\therefore\ 4+2(-1)=2a \ \Rightarrow\ a=1$

Hence the required equation of line is, $x+2y=2$

**31. (d) :** Parallel to $x$-axis $\Rightarrow$ $l$ must be zero.

**32. (a) :** Let $L_1 \equiv 2x+3y-7=0$ and

$L_2 \equiv 2x+3y-5=0$

Here slope of $L_1$ = slope of $L_2$ = $-2/3$

Hence the lines are parallel.

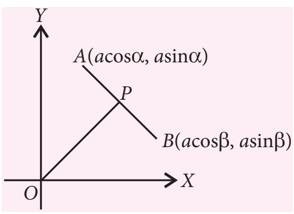
**33. (c) :** Gradient of the line which passes through (1, 0) and $(-2,\sqrt{3})$ is

$$m=\frac{\sqrt{3}-0}{-2-1}=-\frac{1}{\sqrt{3}}\ \Rightarrow\ \theta=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=150^\circ$$

**34. (b) :** $\theta=90^\circ-\tan^{-1}\left(\frac{1}{3}\right)$

$$\Rightarrow\ \tan\theta=\cot\left[\tan^{-1}\left(\frac{1}{3}\right)\right]=3\ \Rightarrow\ \theta=\tan^{-1}(3)$$

**35. (a) :** Mid point of ($a\cos\alpha$, $a\sin\alpha$) and ($a\cos\beta$, $a\sin\beta$) is $\left(\frac{a(\cos\alpha+\cos\beta)}{2},\frac{a(\sin\alpha+\sin\beta)}{2}\right)$




Slope of line $AB$ is

$$\frac{a\sin\beta-a\sin\alpha}{a\cos\beta-a\cos\alpha}=\frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}=m_1$$

And slope of line $OP$ is $\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=m_2$

Now, $m_1\times m_2=\frac{\sin^2\beta-\sin^2\alpha}{\cos^2\beta-\cos^2\alpha}=-1$

Hence the lines are perpendicular.

**36. (c) :** $P$ is centroid of $\Delta ABC$

$\therefore$ Area of $\Delta ABC = 6\times 5 = 30$ sq. units

**37. (a, d) :** Note that lines $u=0$, $v=0$ are perpendicular. Make the co-ordinate axes coincide with $u=0$, $v=0$. Now the lines $L_1\equiv 0$, $L_2\equiv 0$ are equally inclined with $uv$ axes.

$\therefore\ u=0,\ v=0$ are bisectors.

**38. (a, b, c) :** Inclinations of two lines are $\theta$ and $\phi$

$\therefore$ Inclination of angle bisector is $\frac{\theta+\phi}{2}$

$\Rightarrow\ \alpha=\frac{\theta+\phi}{2}$ and $\tan\alpha\times\frac{\gamma}{\beta}=-1 \Rightarrow \tan\alpha=\frac{-\beta}{\gamma}$

$\therefore\ \beta=-\sin\alpha,\ \gamma=\cos\alpha\ \Rightarrow\ \beta^2+\gamma^2=1$

**39. (b, c) :** (3, 2), (–4, 1), (–5, 8) form a right angled triangle at (–4, 1).

Orthocentre is (–4, 1), circumcentre is mid point of (3, 2) and (–5, 8) is (–1, 5)

**40. (a, d) :** Dividing point of $P(-5, 1)$, $Q(3, 5)$ in the ratio $k$ : 1 is

$A\left(\frac{3k-5}{k+1},\frac{5k+1}{k+1}\right)$, $B$ (1, 5), $C$ (7, –2)

Area of triangle $ABC$ = 2

On solving, $k=7,\ \frac{31}{9}$

**41. (a, b, c, d) :** Equation of the curve passing through all four points $A$, $B$, $C$, $D$ can be written as

$(3x+4y-24)(4x+3y-24)+\lambda xy=0$

Now for different values of $\lambda$ we will get different curves.

**42. (a, b) :** $(2a+3b-c)(3a-b+c)=0$

$\Rightarrow\ -2a-3b+c=0$ or $3a-b+c=0$

**43. (b) :** The quadrilateral formed by angular bisectors is a rectangle whose sides are

$$|(a-b)|\sin\frac{\alpha}{2},\ |(a-b)|\cos\frac{\alpha}{2}$$

$S = ab\sin\alpha$

$$Q=\frac{1}{2}(a-b)^2\sin\alpha$$

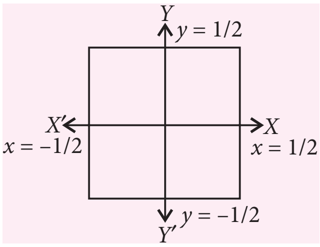
**44. (c) :** $\dfrac{S}{Q}=\dfrac{2ab}{(a-b)^2} \Rightarrow \dfrac{a}{b}=\dfrac{S+Q\pm\sqrt{Q^2+2QS}}{S}$

**45. (a) :** Note that sides are $(a-b)\sin\dfrac{\alpha}{2}$ and $(a-b)\cos\dfrac{\alpha}{2}$

**46. A → q ; B → p, q, r ; C → r**

**(A)** $\because$ $\max\{|x|,|y|\}=1/2$

$\begin{cases}|x|=1/2, & \text{if } |y|<1/2\\ |y|=1/2, & \text{if } |x|<1/2\end{cases}$




Required area = 1 × 1

= 1 sq. unit

**(B)** The line $y = x$ cuts the lines $|x + y| = 6$

*i.e,* $x + y = \pm 6$

At $x = \pm 3, \Rightarrow (x, y) \equiv (-3, -3)$ and $(3, 3)$

then $-3 < a < 3$

$\therefore$ $0 \le |a| < 3$

$\Rightarrow$ $[|a|] = 0, 1, 2$

**(C)** Since (0, 0) and (1, 1) lie on the same side.

So, $a^2 + ab + 1 > 0$

$\because$ Coefficient of $a^2 > 0$ $\therefore$ $D < 0$

$b^2 - 4 < 0$ or $-2 < b < 2$

$\Rightarrow$ $b = -1, 0, 1$

$\therefore$ Number of non-zero integral values of $b$ are 2. $(b = -1$ and $b = 1)$

**47. (2) :** If $x \in (0, 1)$

Then $-1 \le x + y < 0$

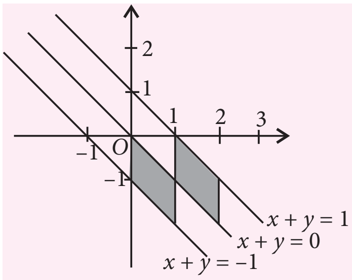
And if $x \in [1, 2)$

$0 \le x + y < 1$

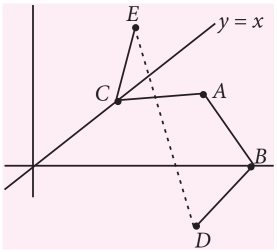
Required area

$= 4\left(\dfrac{1}{2}\cdot 1\cdot\sqrt{2}\sin\dfrac{\pi}{4}\right)$

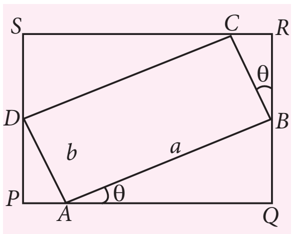
= 2 sq. units




**48. (3) :** Let, $D = (2, -1)$ be the reflection of $A$ in $x$-axis, and let $E = (1, 2)$ be the reflection in the line $y = x$. Then $AB = BD$ and $AC = CE$, so the perimeter of $ABC$ is $DB + BC + CE \ge DE = \sqrt{10}$




**49. (4) :** $2(a + b) = x$ (a constant)




Area of $PQRS = (b\sin\theta + a\cos\theta)(a\sin\theta + b\cos\theta)$

$= ab+\dfrac{a^2+b^2}{2}\sin 2\theta \le \dfrac{(a+b)^2}{2}=\dfrac{x^2}{8}$ $\therefore$ $\dfrac{x^2}{8}=32 \Rightarrow x=16$

**50. (1)**

◈◈

# BRAIN @ WORK

## PERMUTATIONS & COMBINATIONS

This article is a collection of shortcut methods, important formulas and MCQs along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PETs.

### FUNDAMENTAL PRINCIPLES OF COUNTING

- **Multiplication Principle**

If an operation $A$ can be performed in '$m$' different ways and a second operation $B$ can be performed in '$n$' different ways and $C$ is a work which is done only when both $A$ and $B$ are done, then the number of ways of doing the work $C$ is $m \times n$. This can be extended to any finite number of operations.

If there are $n$ jobs $J_1, J_2, ..., J_n$ such that job $J_i$ can be performed independently in $m_i$ ways ; $i = 1, 2, ...., n$ and there is work $C$ which is done only when all the works $(J_1, J_2, ..., J_n)$ are done. Then the number of ways of doing the work $C$ is $m_1 \times m_2 \times m_3 \times........\times m_n$.

- **Addition Principle**

If an operation $A$ can be performed in '$m$' different ways and another operation $B$, which is independent of the first operation, can be performed in '$n$' ways and $C$ is a work which is done only when either $A$ or $B$ is done, then the number of ways of doing the work $C$ is $(m + n)$, this can be extended to any infinite number of mutually exclusive operations.

### PERMUTATIONS

Each of the different arrangements which can be made by taking some or all of a number of distinct objects at a time is called a permutation. Permutation of things means arrangement of things. The word arrangement is used if order of things is taken into account. Thus if order of different things changes, then their arrangement also changes.

### Theorems

- The number of permutations of $n$ different things, taken $r$ at a time (repetition not allowed) is denoted by ${}^nP_r$, where ${}^nP_r$ is defined as $\dfrac{n!}{(n-r)!}$.
- The number of permutations of $n$ different things taken all at a time (repetition not allowed) = $n!$.
- The number of permutations of $n$ different things, taken $r$ at a time when each thing may be repeated any number of times is $n^r$.
- Total number of arrangements of the $n$ objects taken all at a time when each thing may be repeated any number of times is $n^n$.
- The number of permutations of $n$ things taken all at a time where $p$ are alike of one kind, $q$ are alike of second kind, $r$ are alike of third kind and rest all are different is $\dfrac{n!}{p!q!r!}$.

### COMBINATIONS

Each of the different groups or selections which can be made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed, is called a combination. Combination means selection only and permutation means selection + arrangement.

**Theorem :** The number of combinations of $n$ different things, taken $r$ at a time (repetition not allowed) is denoted by ${}^nC_r$ where ${}^nC_r = \dfrac{n!}{r!(n-r)!}, 0 \le r \le n$.

### CONDITIONAL COMBINATION

- Number of ways of choosing $r$ things out of $n$ given things if $p$ particular things must be excluded is ${}^{(n-p)}C_r$.

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- Number of ways of choosing $r$ things out of $n$ given things if $p$ particular things must be included $(p \le r)$ is ${}^{n-p}C_{r-p}$.

## CONDITIONAL PERMUTATIONS

- The number of all permutations (arrangements) of $n$ different objects taken $r$ at a time,
  (a) When a particular object is to be always included in each arrangement is ${}^{n-1}P_{r-1} \cdot r$
  (b) When a particular object is never taken in each arrangement is ${}^{n-1}P_r$
- When all of a certain given things are not to occur together :
  In order to find the number of permutations when all of a certain given things are not to occur together, find
  (a) the total number of arrangements when there is no restriction. Let this number be $x$.
  (b) number of arrangements when all of the things (which are not to occur together) are together. Let this number by $y$.
  (c) Required number = $x - y$.
- When no two of a certain given things occur together :
  In order to find the number of permutations when no two of a certain given things occur together.
  (a) First of all put the $m$ things on which there is no restriction in a line. These $m$ things can be arranged in $m!$ ways.
  (b) Then count the number of places between every two of $m$ things on which there is no restriction including end positions. Number of such places will be $(m + 1)$.
  (c) If $m$ is the number of things on which there is no restriction and $n$ is the number of things no two of which are to occur together, then required number of ways = ${}^{m+1}P_n \times m!$.
- If two type of things are to be arranged alternately, then
  (a) if there numbers differ by 1 put the thing whose number is greater at first, third, fifth ... places etc. and other things at second, fourth, sixth .... places. Let the number of numbers be $m + 1$ and $m$ then the required number of ways = $(m + 1)! \times m!$.
  (b) If the number of two types of things is same, consider two cases separately keeping first type of things at first place, third, fifth place... etc. and second type of things at first, third, fifth place ... and then add. Let the number of things be $m$, then the required number of ways = $2 \times m! \times m!$.

### Selection of One or More Objects

- Selection from Different Objects
  (a) The number of ways of selecting any number of objects out of $n$ different objects = $2^n$
  (b) The number of ways of selecting atleast one object out of $n$ different objects
  $= {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$
- Selection from Identical Objects
  (a) The number of ways of selecting $r$ objects out of $n$ identical objects is 1.
  (b) The number of ways of selecting any number (zero or more) of objects out of $n$ identical objects is $n + 1$.
  (c) The total number of selections of some or all out of $x + y + z$ items where $x$ are alike of one kind, $y$ are alike of second kind and $z$ are alike of third kind is $(x + 1)(y + 1)(z + 1)$.
  (d) The total number of selections of atleast one out of $x + y + z$ items where $x$ are alike of one kind, $y$ are alike of second kind and $z$ are alike of third kind is $[(x + 1)(y + 1)(z + 1)] - 1$.
- Selection from Identical and Distinct Objects
  (a) If we have $x$ alike objects of one kind, $y$ alike objects of second kind, $z$ alike objects of third kind and $k$ different objects, then the number of ways of selecting any number of objects $= (x + 1)(y + 1)(z + 1)\cdot 2^k$
  (b) If we have $x$ alike objects of one kind, $y$ alike objects of second kind, $z$ alike objects of third kind and $k$ different objects, then the number of ways of selecting atleast one object $= \{(x + 1)(y + 1)(z + 1)\cdot 2^k\} - 1$

## DIVISION AND DISTRIBUTION OF DISTINCT OBJECTS

- The number of ways of dividing $n$ distinct objects in $r$ groups of different sizes containing $a_1, a_2, a_3, ..., a_r$ objects respectively, where $n = a_1 + a_2 + a_3 + ... + a_r$ and $a_1, a_2, a_3, ..., a_r$ are all different numbers
  $$= {}^nC_{a_1}\, {}^{n-a_1}C_{a_2}\, {}^{n-a_1-a_2}C_{a_3} \ldots {}^{a_r}C_{a_r} = \frac{n!}{a_1!a_2!a_3!\cdots a_r!}$$
- The number of ways of distributing $n$ distinct objects among $r$ people such that one of them gets $a_1$ objects, some one gets $a_2$, some one gets $a_3$, ... and some one gets $a_r$ where $a_1 + a_2 + ... + a_r = n$ and $a_1, a_2, a_3, ..., a_r$ are different numbers $= \dfrac{n!}{a_1!a_2!a_3!\cdots a_r!}.r!$

- The number of ways in which $m + n$ things can be divided into two groups containing $m$ and $n$ things respectively $= \frac{(m+n)!}{m!\,n!}$.
- If $n = m$, the groups are equal, and in this case the number of different ways of subdivision $= \frac{2m!}{m!m!2!}$
- If $2m$ things are to be divided equally between two persons, then the number of divisions $= \frac{2m!}{m!m!}$.
- The number of divisions of $m + n + p$ things into groups of $m$, $n$ and $p$ things respectively $= \frac{(m+n+p)!}{m!n!p!}$
- If $3m$ things are divided into three equal groups, then the number of divisions $= \frac{(3m)!}{m!m!m!3!}$.
- If $3m$ things are to be divided among three persons, then the number of divisions $= \frac{(3m)!}{m!m!m!}$.

## COMBINATION WITH REPETITION

- If there are $a_1$ objects of I$^{st}$ kind, $a_2$ objects of 2$^{nd}$ kind, $a_3$ objects of 3$^{rd}$ kind, ..., $a_n$ objects of $n^{th}$ kind and we want to choose $r$ objects out of these under the condition that at least one object of every kind should be chosen.
  Number of ways = Coefficient of $x^r$ in
  $$(x+x^2+...+x^{a_1})(x+x^2+...+x^{a_2})...(x+x^2+...+x^{a_n})$$
- Number of ways in which $r$ identical things can be distributed among $n$ persons when each person can get zero or more things
  = Coeff. of $x^r$ in $(1 + x + x^2 + ... + x^r)^n$
  = Coeff. of $x^r$ in $\left(\frac{1-x^{r+1}}{1-x}\right)^n$
  = Coeff. of $x^r$ in $[(1 - x^{r+1})^n (1 - x)^{-n}]$
  = Coeff. of $x^r$ in $(1 - x)^{-n}$ [leaving terms containing powers of $x$ greater than $r$]
  $= {}^{n+r-1}C_r$

## NUMBER OF INTEGRAL SOLUTIONS OF LINEAR EQUATIONS AND INEQUATIONS

- Consider the equation
  $x_1 + x_2 + x_3 + x_4 + .... + x_r = n$ ....(1)
  where $x_1, x_2, x_3, x_4, ..., x_r$ and $n$ are non-negative integers.
  This equation may be interpreted as that $n$ identical objects are to be divided into $r$ groups where a group may contain any number of objects. Therefore,
  Total number of solutions of equation (1)
  = Coefficient of $x^n$ in $(x^0 + x^1 + ... + x^n)^r$
  $= {}^{n+r-1}C_r$ or ${}^{n+r-1}C_{n-1}$.
- Total number of solution of the equation (1) when $x_1, x_2, x_3 ..., x_r$ are natural number
  = Coefficient of $x^n$ in $(x^1 + x^2 + x^3 +.... + x^n)^r$
  $= {}^{n-1}C_{r-1}$
- Consider the equation
  $x_1 + 2x_2 + 3x_3 + ... + qx_r = n$
  where $x_1, x_2, x_3 ....., x_r$ and $n$ are non-negative integers then total number of solution of the above equation is = Coefficient of $x^n$ in $(1 + x + x^2 + ... )(1 + x^2 + x^4 + ... ) (1 + x^3 + x^6 + ... ) ... (1 + x^q + x^{2q} + ...)$
  If zero is excluded, then the number of solutions of the above equation = Coefficient of $x^n$ in $(x + x^2 + x^3 + ... ) (x^2 + x^4 + x^6 + ... ) (x^3 + x^6 + x^9 + ... ) ... (x^q + x^{2q} + ... )$

**Note :**

(i) The coefficient of $x^r$ in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$.

(ii) The coefficient of $x^r$ in $(1 + x)^{-n}$ is $(-1)^r\, {}^{n+r-1}C_r$.

## CIRCULAR PERMUTATIONS

- If clockwise and anticlockwise orders are taken as different, number of circular arrangements of $n$ different things taken all at a time = $(n - 1)!$
  If clockwise and anticlockwise orders are not taken as different, number of circular arrangements of $n$ different things taken all at a time $= \frac{1}{2}(n-1)!$
- If a circular arrangement can be flipped or turned upside down, number of circular arrangements of $n$ different things taken all at a time $= \frac{(n-1)!}{2}$, otherwise it is $(n - 1)!$
- If positions in a circular arrangement are numbered, number of circular arrangements of $n$ different things taken all at a time = $n!$, otherwise obviously $(n - 1)!$
- Number of circular permutations of $n$ different things taken $r$ at a time if clockwise and anticlockwise orders are taken as different $= {}^nC_r \cdot (r - 1)!$
- Number of circular permutations of $n$ different things taken $r$ at a time if clockwise and anticlockwise orders are not taken as different $= {}^nC_r\, \frac{(r-1)!}{2}$

## DERANGEMENTS

Derangement means destroy the arrangement *i.e.*, rearranging the objects in such a way that no object remains at its original place.

- If $n$ distinct things are arranged in a row, then number of ways in which they can be deranged such that none of them occupies its original place $= n! - {}^nC_1\,(n-1)! + {}^nC_2\,(n-2)! - {}^nC_3\,(n-3)! + \ldots + (-1)^n\,{}^nC_n \cdot 0!$
$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$
- If $r(0 \le r \le n)$ objects occupy the places assigned to them *i.e.*, their original places and none of the remaining $(n-r)$ objects occupies its original place, then the number of such ways
$$= {}^nC_r \cdot (n-r)!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right\}$$

## PROBLEMS

**1.** $n_1$ and $n_2$ are four digit numbers. Total number of ways of forming $n_1$ and $n_2$ so that $n_2$ can be subtracted from $n_1$ without borrowing at any stage, is equal to

(a) $(36)\,(55)^3$ (b) $(45)\,(55)^3$
(c) $(55)^4$ (d) None of these

**2.** Total number of positive integral solutions of $15 < x_1 + x_2 + x_3 \le 20$ is equal to

(a) 1125 (b) 1150
(c) 1245 (d) 685

**3.** '$n$' is selected from the set {1, 2, 3, ....., 100} and the number $2^n + 3^n + 5^n$ is formed. Total number of ways of selecting '$n$' so that the formed number is divisible by 4, is equal to

(a) 50 (b) 49
(c) 48 (d) none of these

**4.** Total number of times, the digit '3' will be written, when the integers having less than 4 digits are listed, is equal to.

(a) 300 (b) 271
(c) 298 (d) none of these

**5.** A variable name in certain computer language must be either a alphabet or alphabet followed by a digit. Total number of different variable names that can exist in that language is equal to

(a) 280 (b) 240
(c) 286 (d) 80

**6.** Total number of ways of selecting two numbers from the set {1, 2, 3, 4, ......., $3n$} so that their sum is divisible by 3, is equal to

(a) $\dfrac{2n^2 - n}{2}$ (b) $\dfrac{3n^2 - n}{2}$
(c) $2n^2 - n$ (d) $3n^2 - n$

**7.** The total number of ways of selecting 10 balls out of an unlimited number of identical white, red and blue balls, is equal to

(a) ${}^{12}C_2$ (b) ${}^{12}C_3$
(c) ${}^{10}C_2$ (d) ${}^{10}C_3$

**8.** There are 10 persons among whom two are brothers. The total number of ways in which these persons can be seated around a round table so that exactly one person sits between the brothers, is equal to

(a) (2!) (7!) (b) (2!) (8!)
(c) (3!) (7!) (d) (3!) (8!)

**9.** A library has '$a$' copies of one book, '$b$' copies each of two books, '$c$' copies each of three books and single copy of '$d$' books. The total number of ways in which these books can be arranged in a shelf, is equal to

(a) $\dfrac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$ (b) $\dfrac{(a+2b+3c+d)!}{a!(2b!)(3c!)^3}$
(c) $\dfrac{(a+2b+3c+d)!}{(c!)^3}$ (d) $\dfrac{(a+2b+3c+d)!}{a!(2b!)(c!)}$

**10.** The number of numbers that are less than 1000 that can be formed using the digits 0, 1, 2, 3, 4, 5 such that no digit is being repeated in the formed number, is equal to

(a) 130 (b) 131 (c) 156 (d) 155

**11.** Total number of six digit numbers that can be formed, having the property that every succeeding digit is greater than the preceding digit, is equal to

(a) ${}^9C_3$ (b) ${}^{10}C_3$ (c) ${}^9P_3$ (d) ${}^{10}P_3$

**12.** If letters of the word 'KUBER' are written in all possible orders and arranged as in a dictionary, then rank of the word 'KUBER' will be

(a) 67 (b) 68 (c) 65 (d) 69

**13.** In a chess tournament, all participants were to play one game with the other. Two players fell ill after having played 3 games each. If total number of games played in the tournament is equal to 84, then total number of participants in the beginning was equal to

(a) 10 (b) 15 (c) 12 (d) 14

**14.** The total number of flags with three horizontal strips, in order, that can be formed using 2 identical red, 2 identical green and 2 identical white strips, is equal to
(a) 4!  (b) 3·(4!)
(c) 2·(4!)  (d) none of these

**15.** The sides $AB$, $BC$, $CA$ of a triangle $ABC$ have 3, 4, 5 interior points respectively on them. Total number of triangles that can be formed using these points as vertices, is equal to
(a) 135  (b) 145  (c) 178  (d) 205

**16.** '$n_1$' men and '$n_2$' women are to be seated in a row so that no two women sit together. If $n_1 > n_2$, then total number of ways in which they can be seated, is equal to
(a) ${}^{n_1}C_{n_2}$  (b) ${}^{n_1}C_{n_2}(n_1!)(n_2!)$
(c) ${}^{n_1}C_{n_2+1}(n_1!)(n_2!)$  (d) ${}^{n_1+1}C_{n_2}(n_1!)(n_2!)$

**17.** There are '$n$' numbered seats around a round table. Total number of ways in which $n_1(n_1 < n)$ persons can sit around the table, is equal to
(a) ${}^{n}C_{n_1}$  (b) ${}^{n}P_{n_1}$
(c) ${}^{n}C_{n_1-1}$  (d) ${}^{n}P_{n_1-1}$

**18.** Three boys of class X, 4 boys of class XI and 5 boys of class XII, sit in a row. Total number of ways in which these boys can sit so that all the boys of same class sit together, is equal to
(a) $(3!)^2\,(4!)\,(5!)$  (b) $(3!)\,(4!)^2\,(5!)$
(c) $(3!)\,(4!)\,(5!)$  (d) $(3!)\,(4!)\,(5!)^2$

**19.** Total number of ways in which the letters of the word 'MISSISSIPPI' be arranged, so that any two S's are separated, is equal to
(a) 7350  (b) 3675
(c) 6300  (d) none of these

**20.** The number of ways in which a mixed double game can be arranged amongst nine married couples so that no husband and his wife play in the same game, is equal to
(a) ${}^{9}C_2 \cdot {}^{7}C_2$  (b) ${}^{9}C_2 \cdot {}^{7}C_2 \cdot {}^{2}C_1$
(c) ${}^{9}P_2 \cdot {}^{7}P_2$  (d) ${}^{9}P_2 \cdot {}^{7}P_2 \cdot {}^{2}P_1$

**21.** A candidate is required to answer 7 out of 10 questions, which are divided into two groups, each containing 5 questions. He is not permitted to attempt more than 4 questions from each group. Total number of different ways in which the candidate can answer the paper, is equal to
(a) $2 \cdot {}^{5}C_3 \cdot {}^{5}C_4$  (b) $2 \,.\, {}^{5}P_3 \cdot {}^{5}P_4$
(c) ${}^{5}C_3 \cdot {}^{5}C_4$  (d) ${}^{5}P_3 \cdot {}^{5}P_4$

**22.** The total number of six digit numbers $x_1\, x_2\, x_3\, x_4\, x_5\, x_6$ having the property that $x_1 < x_2 \le x_3 < x_4 < x_5 \le x_6$ is equal to
(a) ${}^{10}C_6$  (b) ${}^{12}C_6$
(c) ${}^{11}C_6$  (d) none of these

**23.** The total number of three digit numbers, the sum of whose digits is even, is equal to
(a) 450  (b) 350  (c) 250  (d) 325

**24.** '$n$' different toys have to be distributed among '$n$' children. Total number of ways in which these toys can be distributed so that exactly one child gets no toy, is equal to
(a) $n!$  (b) $n!\,{}^{n}C_2$
(c) $(n-1)!\,{}^{n}C_2$  (d) $n!\,{}^{n-1}C_2$

**25.** Total number of permutations of '$k$' different things, in a row, taken not more than '$r$' at a time (each thing may be repeated any number of times) is equal to
(a) $k^r - 1$  (b) $k^r$
(c) $\dfrac{k^r - 1}{k-1}$  (d) $\dfrac{k(k^r - 1)}{(k-1)}$

**26.** Total number of 4 digit numbers that are greater than 3000 and can be formed using the digits 1, 2, 3, 4, 5, 6 (no digit is being repeated in any number ) is equal to
(a) 120  (b) 240  (c) 480  (d) 80

**27.** A teacher takes 3 children from her class to the zoo at a time as often as she can, but she doesn't take the same set of three children more than once. She finds out that she goes to the zoo 84 times more than a particular child goes to the zoo. Total number of students in her class in equal to
(a) 12  (b) 14  (c) 10  (d) 11

**28.** A person predicts the outcome of 20 cricket matches of his home team. Each match can result either in a win, loss or tie for the home team. Total number of ways in which he can make the predictions so that exactly 10 predictions are correct, is equal to
(a) ${}^{20}C_{10} \cdot 2^{10}$  (b) ${}^{20}C_{10} \cdot 3^{20}$
(c) ${}^{20}C_{10} \cdot 3^{10}$  (d) ${}^{20}C_{10} \cdot 2^{20}$

**29.** A team of four students is to be selected from a total of 12 students. Total number of ways in which team can be selected such that two particular students refuse to be together and other two particular students wish to be together only, is equal to
(a) 220  (b) 182
(c) 226  (d) none of these

**30.** Two players $P_1$ and $P_2$ play a series of '$2n$' games. Each game can result in either a win or loss for $P_1$. Total number of ways in which $P_1$ can win the series of these games, is equal to

(a) $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ (b) $\frac{1}{2}(2^{2n} - 2 \cdot {}^{2n}C_n)$

(c) $\frac{1}{2}(2^{n} - {}^{2n}C_n)$ (d) $\frac{1}{2}(2^{n} - 2 \cdot {}^{2n}C_n)$

**31.** Total number of 3 letter words that can be formed from the letters of the word 'SAHARANPUR', is equal to

(a) 210 (b) 237 (c) 247 (d) 227

**32.** 15 identical balls have to be put in 5 different boxes. Each box can contain any number of balls. Total number of ways of putting the balls into box so that each box contains atleast 2 balls, is equal to

(a) ${}^9C_5$ (b) ${}^{10}C_5$ (c) ${}^6C_5$ (d) ${}^{10}C_6$

**33.** Total number of positive integral solutions of the equation $x_1 \cdot x_2 \cdot x_3 = 60$ is equal to

(a) 27 (b) 54

(c) 64 (d) none of these

**34.** Total number of four digit numbers having all different digits, is equal to

(a) 4536 (b) 504 (c) 5040 (d) 720

**35.** Total number of 5 digit numbers having all different digits and divisible by 4 that can be formed using the digits {1, 3, 2, 6, 8, 9}, is equal to

(a) 192 (b) 32 (c) 1152 (d) 384

## SOLUTIONS

**1.** **(b) :** Let $n_1 = x_1\,x_2\,x_3\,x_4$ and $n_2 = y_1\,y_2\,y_3\,y_4$

$n_2$ can be subtracted from $n_1$ without borrowing at any stage if $x_i \ge y_i$

For $i = 2, 3, 4$; let $x_i = r(r = 0, 1, 2, ...,9)$

$\Rightarrow\ y_i \le r \Rightarrow y_i = 0, 1, 2, ..., r$

That mean $y_i$ can be selected in $(r + 1)$ ways.

Thus, total ways of selecting $x_i$ and $y_i$ suitably

$$= \sum_{r=0}^{9}(r+1) = 1 + 2 + 3 + ..... + 10 = \frac{11\cdot 10}{2} = 55$$

For $i = 1$, let $x_i \le r(r = 1, 2, ...., 9)$

$\Rightarrow y_i \le r \Rightarrow y_i = 1, 2, ...., r$

That means $y_i$ can be selected in '$r$' ways.

Thus, total ways of selecting $x_1$, $y_1$ suitably

$$= \sum_{r=1}^{9} r = 1 + 2 + .... + 9 = \frac{9\cdot 10}{2} = 45$$

Thus, total ways $= 45(55)^3$

**2.** **(d) :** $15 < x_1 + x_2 + x_3 \le 20$

$\Rightarrow\ x_1 + x_2 + x_3 = 16 + r,\ r = 0, 1, 2, 3, 4.$

Now number of positive integral solutions of

$x_1 + x_2 + x_3 = 16 + r$ is ${}^{13+r+3-1}C_{13+r}$,

*i.e.* ${}^{15+r}C_{13+r} = {}^{15+r}C_2$

Thus required number of solutions

$$= \sum_{r=0}^{4} {}^{15+r}C_2 = {}^{15}C_2 + {}^{16}C_2 + {}^{17}C_2 + {}^{18}C_2 + {}^{19}C_2 = 685$$

**3.** **(b) :** If $n$ is odd, then

$3^n = 4\lambda_1 - 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow 2^n + 3^n + 5^n$ is divisible by 4 if $n > 2$.

Thus $n = 3, 5, 7, 9, ........, 99$ *i.e.*, $n$ can take 49 different values.

If $n$ is even, then

$3^n = 4\lambda_1 + 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow\ 2^n + 3^n + 5^n$ is not divisible by 4, as

$2^n + 3^n + 5^n$ will be in the form of $4\lambda + 2$.

Thus, total number of ways of selecting '$n$' = 49.

**4.** **(c) :** Any integer having less than 4 digits will be in the form of $xyz$.

If 3 is used exactly once, then the number of ways $= {}^3C_1 \cdot 9^2$

If 3 is used exactly 2 times, then the number of ways $= ({}^3C_2 \cdot 9) \cdot 2$

If 3 is used exactly 3 times, then there is only one such number.

Thus, required number of ways

$= 1 + 2 \cdot {}^3C_2 \cdot 9 + {}^3C_1 \,.\, 9^2 = 298$

**5.** **(c) :** Total variables if only the alphabet is used is equal to 26.

Total variables if alphabets and digits both are used = $26 \cdot 10$.

$\therefore$ Total variables $= 26(1 + 10) = 286$

**6.** **(b) :** Given numbers can be rearranged as

1 4 7 .... $3n - 2 \longrightarrow 3\lambda - 2$ type

2 5 8 .... $3n - 1 \longrightarrow 3\lambda - 1$ type

3 6 9 .... $3n \longrightarrow 3\lambda$ type

That means we must take two numbers from last or one number each from first and second row.

Total number of ways $= {}^nC_2 + {}^nC_1 \cdot {}^nC_1$

$$= \frac{n(n-1)}{2} + n^2 = \frac{3n^2 - n}{2}$$

**7.** **(a) :** Let $x_W$, $x_R$, $x_B$ be the number of white balls, red balls and blue balls respectively being selected.

We must have

$$x_W + x_R + x_B = 10.$$

Required number of ways

= Number of non-negative integral solutions of

$x_W + x_R + x_B = 10$

$= {}^{3+10-1}C_{10} = {}^{12}C_{10} = {}^{12}C_2$

**8. (b) :** Person who has to sit between the brothers can be selected in ${}^8C_1$ ways.

Thus, total ways = $8\cdot(7!)\cdot(2!) = (8!)\cdot(2!)$

**9. (a) :** Total number of books = $a + 2b + 3c + d$

Total number of ways in which these books can be arranged in a shelf (in same row)

$$= \frac{(a+2b+3c+d)!}{a!(b!)^2\cdot(c!)^3}$$

**10. (b) :** That means formed number can be atmost of three digits.

Total number of one digit numbers = 6

Total number of two digit numbers = $5\cdot5 = 25$

Total number of three digit numbers= $5\cdot5\cdot4 = 100$

Thus, total number of such numbers = 131

**11. (a) :** Let the number is $x_1\,x_2\,x_3\,x_4\,x_5\,x_6$.

We must have

$$x_1 < x_2 < x_3 < x_4 < x_5 < x_6.$$

Clearly no digit can be zero.

Thus total number of such numbers = ${}^9C_6 = {}^9C_3$.

**12. (a) :** Alphabetical order of these letters is B, E, K, R, U.

Total words starting with B = 4! = 24

Total words starting with E = 4! = 24

Total words starting with KB = 3! = 6

Total words starting with KE = 3! = 6

Total words starting with KR = 3! = 6

Next word will be KUBER = 24 + 24 + 18 + 1 = 67

**13. (b) :** Let there were '$n$' players in the beginning.

Total number of games to be played was equal to ${}^nC_2$ and each player would have played $(n-1)$ games. Thus

${}^nC_2 - ((n-1) + (n-1) - 1) + 6 = 84$

$\Rightarrow \quad n^2 - 5n - 150 = 0$

$\Rightarrow \quad n = 15$

**14. (a) :** All strips are of different colours, then number of flags = 3! =6

When two strips are of same colour, then

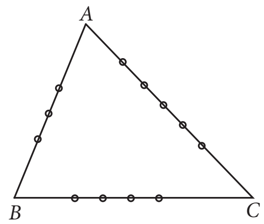
number of flags = ${}^3C_1\cdot\frac{3!}{2}\cdot{}^2C_1 = 18$

$\therefore$ Total flags = 6 + 18 = 24 = 4!

**15. (d) :** Total number of triangles

$= {}^3C_1\cdot{}^4C_1\cdot{}^5C_1 + {}^3C_2({}^4C_1 + {}^5C_1)$
$\quad + {}^4C_2({}^3C_1 + {}^5C_1) + {}^5C_2({}^3C_1 + {}^4C_1)$

$= 3\cdot4\cdot5 + 3\cdot9 + 6\cdot8 + 10\cdot7 = 205$




**16. (d) :** There will be $(n_1 + 1)$ gaps created by $n_1$ men.

Now women have to be seated only in these gaps.

Thus number of such sitting arrangements

$$= {}^{n_1+1}C_{n_2}\cdot n_1!\cdot n_2!$$

**17. (b) :** When seats are numbered, circular permutation is same as linear permutation. Thus total number of sitting arrangements is equal to ${}^nP_{n_1}$.

**18. (a) :** We can think of three groups. One consisting of three boys of class X, other consisting of 4 boys of class XI and last one consisting of 5 boys of class XII. These groups can be arranged in 3! ways and boys in these groups can be further arranged in 3!, 4! and 5! ways respectively.

Thus total ways = $(3!)^2\,(4!)\,(5!)$

**19. (a) :** 1M, 4I, 2P and 4S. Total gaps created by letter 1M, 4I and 2P is 8.

Thus total arrangements = ${}^8C_4\cdot\frac{7!}{2!4!} = 7350$

**20. (b) :** First of all two men can be selected in ${}^9C_2$ ways. Thereafter 2 women can selected in ${}^7C_2$ ways (as wives of selected men are not be selected). And finally they can be paired up in ${}^2C_1$ ways.

Thus total ways = ${}^9C_2\cdot{}^7C_2\cdot{}^2C_1$.

**21. (a) :** Choices available to the candidates are; 3 questions from first group and 4 from another or 4 questions for first group and 3 questions from another.

Thus total ways = $2\cdot{}^5C_3\cdot{}^5C_4$

**22. (c) :** $x_1 < x_2 \le x_3 < x_4 < x_5 \le x_6$

It will give rise to following four cases :

(i) $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \rightarrow {}^9C_6$ ways

(ii) $x_1 < x_2 = x_3 < x_4 < x_5 < x_6 \rightarrow {}^9C_5$ ways

(iii) $x_1 < x_2 < x_3 < x_4 < x_5 = x_6 \rightarrow {}^9C_5$ ways

(iv) $x_1 < x_2 = x_3 < x_4 < x_5 = x_6 \rightarrow {}^9C_4$ ways

Thus total number of such numbers

$= {}^9C_6 + {}^9C_5 + {}^9C_5 + {}^9C_4 = {}^{10}C_6 + {}^{10}C_5 = {}^{11}C_6$

**23. (a) :** Let the number be $n = x_1\, x_2\, x_3$
Since $x_1 + x_2 + x_3$ is even. That means there are following cases :
(i) $x_1$ , $x_2$, $x_3$ all are even $\rightarrow 4 \cdot 5 \cdot 5 = 100$ ways
(ii) $x_1$ is even and $x_2$, $x_3$ are odd $\rightarrow 4 \cdot 5 \cdot 5 = 100$ ways
(iii) $x_1$, $x_2$ are odd and $x_3$ is even $\rightarrow 5 \cdot 5 \cdot 5 = 125$ ways
(iv) $x_1$ is odd, $x_2$ is even and $x_3$ is odd $\rightarrow 5 \cdot 5 \cdot 5 = 125$ ways
Total ways = 100 + 100 + 125 + 125 = 450.

**24. (b) :** In this case, one child gets no toy and one gets 2 toys and all remaining children get one each.
Corresponding number of ways

$$= \frac{n!}{2!(n-2)!} \cdot n! = n! \cdot {}^nC_2$$

**25. (d) :** Total permutations $= 0 + k + k^2 + k^3 + ..... + k^r$

$$= \frac{k(k^r - 1)}{(k-1)}$$

**26. (b) :** Let the formed number is $x_1\, x_2\, x_3\, x_4$
Clearly, $x_1 \geq 3$.
Thus total number of such numbers
$= 4 \cdot 5 \cdot 4 \cdot 3 = 240$

**27. (c) :** Let the number of students be $n$, then total number of times the teacher goes to zoo is equal to ${}^nC_3$ and total number of times a particular student goes to the zoo is equal to ${}^{n-1}C_2$
Thus ${}^nC_3 - {}^{n-1}C_2 = 84$

$$\Rightarrow \frac{n(n-1)(n-2)}{3!} - \frac{(n-1)(n-2)}{2} = 84$$

$\Rightarrow n(n-1)(n-2) - 3(n-1)(n-2) = 504$
$\Rightarrow (n-1)(n-2)(n-3) = 504$
$\Rightarrow (n-1)(n-2)(n-3) = 9 \cdot 8 \cdot 7$
$\Rightarrow n = 10$

**28. (a) :** Matches whose predictions are correct can be selected in ${}^{20}C_{10}$ ways. Now each wrong prediction can be made in 2 ways.
Thus total ways $= {}^{20}C_{10} \cdot 2^{10}$.

**29. (c) :** Let $S_1$ and $S_2$ refuse to be together and $S_3$ and $S_4$ want to be together only.
Total ways when $S_3$ and $S_4$ are selected
$= ({}^8C_2 + {}^2C_1 \cdot {}^8C_1) = 44$
Total ways when $S_3$ and $S_4$ are not selected
$= ({}^8C_4 + {}^2C_1 \cdot {}^8C_3) = 182$
Thus total ways = 44 + 182 = 226.

**30. (a) :** '$P_1$' must win atleast $(n + 1)$ games.
Let '$P_1$' wins $n + r$ games $(r = 1, 2, ...., n)$
Corresponding ways $= {}^{2n}C_{n+r}$

$$\text{Total ways} = \sum_{r=1}^{n} {}^{2n}C_{n+r}$$

$$= {}^{2n}C_{n+1} + {}^{2n}C_{n+2} + .... + {}^{2n}C_{2n}$$

$$= \frac{2^{2n} - {}^{2n}C_n}{2} = \frac{1}{2}(2^{2n} - {}^{2n}C_n)$$

**31. (c) :** 1S, 3A, 1H, 2R, 1N, 1P, 1U when all letters are different.
Corresponding ways $= {}^7C_3 \cdot 3! = {}^7P_3 = 210$
When two letters are of one kind and other is different.

$$\text{Corresponding ways} = {}^2C_1 \cdot {}^6C_1 \cdot \frac{3!}{2!} = 36$$

When all letters are alike, corresponding ways
$= 210 + 36 + 1 = 247$

**32. (a) :** Let the balls put in the box are $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$.
We must have
$x_1 + x_2 + x_3 + x_4 + x_5 = 15,\ x_i \geq 2$
$\Rightarrow (x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) + (x_5 - 2) = 5$
$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 5,\ y_i = x_i - 2 \geq 0$
Total number of ways is simply equal to number of non-negative integral solutions of the last equation, which is equal to ${}^{5+5-1}C_5 = {}^9C_5$.

**33. (b) :** $x_1 \cdot x_2 \cdot x_3 = 2^2 \cdot 3 \cdot 5$
$\Rightarrow$ Total positive integral solutions = 54

**34. (a) :** Let the number be $x_1 x_2\, x_3\, x_4$.
Then $x_1$ can be chosen in 9 ways. $x_2$ can be chosen in 9 ways. Similarly $x_3$ and $x_4$ can be chosen in 8 and 7 ways respectively.
$\therefore$ Total number of such numbers
$= 9 \cdot 9 \cdot 8 \cdot 7 = 4536$

**35. (a) :** A number is divisible by four, if the last two digits are divisible by four. In this case last two digits can be 12, 16, 28, 32, 36, 68, 92 or 96.
Total number of such numbers $= 8 \cdot ({}^4C_3 \cdot 3!) = 192$.

CLASS XI **Series 6**

# ACE YOUR WAY CBSE

## Sequences and Series

### HIGHLIGHTS

#### SEQUENCE

A sequence is an arrangement of numbers in a definite order according to a certain rule.

**Note :** A sequence may be finite or infinite according to their number of terms.

#### SERIES

Let $\{t_n\}$ be a sequence, then the expression of the form $t_1 + t_2 + .....+ t_n + .....$ is called a series. The series is finite or infinite according to the given sequence is finite or infinite.

#### ARITHMETIC PROGRESSION (A.P.)

An A.P. is a sequence whose terms either increase or decrease by a fixed number.

Such a fixed number is called common difference denoted by '$d$'.

- Let $a$ be the first term, $d$ be the common difference, $l$ be the last term and $n$ be the number of terms in an A.P. such that $a$, $a + d$, $a + 2d$, ...... is an A.P., then

| | |
|---|---|
| General term | $a_n = a + (n - 1)d$ or $l = a + (n - 1)d$ |
| Sum of first $n$ terms | $S_n = \frac{n}{2}\{2a + (n-1)d\}$ or $S_n = \frac{n}{2}(a + l)$ |

We can verify the following simple properties of an A.P.:

- If a constant is added to each term of an A.P., the resulting sequence is also an A.P.
- If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P.
- If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P.
- If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

#### ARITHMETIC MEAN (A.M.)

- Let $a$, $b$ be any two numbers and $A$ be the arithmetic mean between them. Then $a$, $A$, $b$ are in A.P. $\Rightarrow \ A = \frac{a+b}{2}$
- Let $a$, $b$ be any two numbers and $A_1, A_2, ....., A_n$ be $n$ A.M.' s between them. Then $a, A_1, A_2, ..... , A_n, b$ are in A.P. $\Rightarrow \ d = \frac{b-a}{n+1}$

So, $A_1 = a + \frac{b-a}{n+1}$, $A_2 = a + 2\left(\frac{b-a}{n+1}\right)$, .....,

$$A_n = a + n\left(\frac{b-a}{n+1}\right)$$

Sum of $n$ A.M.'s $= n\left(\frac{a+b}{2}\right)$

#### GEOMETRIC PROGRESSION (G.P.)

A G.P. is a succession of numbers in which first term is non-zero and each next term is the product of its preceding term and a non-zero constant.

This non-zero constant is called common ratio denoted by $r$.

- Let $a$ be the first term, $r$ be the common ratio and $n$ be the number of terms such that $a, ar, ar^2, ar^3, .....$ is a G.P., then

| | |
|---|---|
| General term | $a_n = ar^{n-1}$ |
| Sum to $n$ terms | $S_n = \begin{cases} \dfrac{a(1-r^n)}{1-r}, \text{ if } r<1 \\ \dfrac{a(r^n-1)}{r-1}, \text{ if } r>1 \end{cases}$ |
| Sum to infinite terms | $S_\infty = \dfrac{a}{1-r}, \lvert r \rvert < 1$ |

## GEOMETRIC MEAN (G.M.)

- Let $a$, $b$ be any two numbers and $G$ be the geometric mean between them. Then $a$, $G$, $b$ are in G.P. $\Rightarrow \quad G = \sqrt{ab}$
- Let $a$, $b$ be any two numbers and $G_1, G_2, ....., G_n$ be $n$ geometric means between them. Then $a, G_1, G_2, .....,$ $G_n$, $b$ are in G.P.

$$\Rightarrow \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

So, $G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, ....., G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

## RELATIONSHIP BETWEEN A.M. AND G.M.

Let $a$, $b$ be any two numbers and let $A = \dfrac{a+b}{2}$, $G = \sqrt{ab}$

Then, $A \geq G$

## SUM TO $n$ TERMS OF SPECIAL SERIES

- $1+2+3+.....+n = \dfrac{n(n+1)}{2}$
- $1^2+2^2+3^2+.....+n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- $1^3+2^3+3^3+.....+n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

## PROBLEMS

### Very Short Answer Type

**1.** Find the sum of the series
$99 + 95 + 91 + 87 + .....$ to 20 terms.

**2.** Prove that the sum of $n$ arithmetic means between two numbers in $n$ times the single A.M. between them.

**3.** Find the sum of $n$ terms of the series
$(a + b) + (a^2 + 2b) + (a^3 + 3b) + .....$

**4.** If reciprocals of $\dfrac{x+y}{2}$, $y$, $\dfrac{y+z}{2}$ are in A.P., show that $x$, $y$, $z$ are in G.P.

**5.** Find the sum to infinity of the G.P.
$1+\dfrac{1}{3}+\dfrac{1}{9}+\dfrac{1}{27}+.....$

### Short Answer Type

**6.** Between two numbers whose sum is $\dfrac{13}{6}$ an even number of A.M.'s are inserted. If the sum of means exceeds their number by unity, find the number of means.

**7.** If $x$, $y$, $z$ are in A.P. and $A_1$ is the A.M. of $x$ and $y$ and $A_2$ is the A.M. of $y$ and $z$, then prove that the A.M. of $A_1$ and $A_2$ is $y$.

**8.** The sum of three numbers in A.P. is –3 and their product is 8. Find the numbers.

**9.** Find the sum of 10 terms of the G.P. $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}$ .....

**10.** If $x > 0$, prove that $x + \dfrac{1}{x} \geq 2$.

### Long Answer Type - I

**11.** If $a^2 + 2bc$, $b^2 + 2ac$, $c^2 + 2ab$ are in A.P., show that $\dfrac{1}{b-c}, \dfrac{1}{c-a}, \dfrac{1}{a-b}$ are in A.P.

**12.** If $a$, $b$, $c$, $d$ are in G.P., show that
$(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$

**13.** $x + y + z = 15$ if $a$, $x$, $y$, $z$, $b$ are in A.P. and $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{5}{3}$ if $\dfrac{1}{a},\dfrac{1}{x},\dfrac{1}{y},\dfrac{1}{z},\dfrac{1}{b}$ are in A.P., find the G.M. of $a$ and $b$.

**14.** If the sum of $n$ terms of a series is $5n^2 + 3n$, find its $n^{th}$ term. Are the term of this series in A.P. ?

**15.** Find the sum of the series
$1 \cdot n + 2 \cdot (n - 1) + 3 \cdot (n - 2) + ..... + n \cdot 1$

### Long Answer Type - II

**16.** If $a$, $b$ are the roots of equation $x^2 - 3x + p = 0$ and $c$, $d$ are roots of equation $x^2 - 12x + q = 0$, where $a$, $b$, $c$, $d$ forms a G.P., then prove that $(q + p) : (q - p) = 17 : 15$.

**17.** If $a$, $b$, $c$ are in A.P. and $\dfrac{1}{a^2}, \dfrac{1}{b^2}, \dfrac{1}{c^2}$ are in A.P., then prove that either $-\dfrac{a}{2}, b, c$ are in G.P. or $a = b = c$.

18. Find the relation between $x$ and $y$ such that the $r^{\text{th}}$ mean between $x$ and $2y$ may be same as the $r^{\text{th}}$ mean between $2x$ and $y$, if $n$ means are inserted in each case.

19. Natural numbers are divided into groups in the following way : 1;(2, 3); (4, 5, 6); (7, 8, 9, 10)
Show that the sum of the numbers in the $n^{\text{th}}$ group is $\dfrac{n(n^2+1)}{2}$.

20. Find 5 + 7 + 13 + 31 + 85 + ..... to $n$ terms.

## SOLUTIONS

1. The terms of given series are in A.P. whose common difference $d = -4$ and first term $a = 99$.
Now, sum of 20 terms of the series,

$$S_{20} = \frac{20}{2}[2\cdot 99 + (20-1)(-4)]$$
$$= 10\,(198 - 76) = 1220$$

2. Let $A_1, A_2, \ldots, A_n$ be $n$ arithmetic means between $a$ and $b$. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. with common difference $d$ given by $d = \dfrac{b-a}{n+1}$.

Now, $A_1 + A_2 + \ldots + A_n = \dfrac{n}{2}(A_1 + A_n)$

$$= \frac{n}{2}(a+b) = n\left(\frac{a+b}{2}\right) = n \times (\text{A.M. between } a \text{ and } b)$$

3. $(a + b) + (a^2 + 2b) + (a^3 + 3b) + \ldots$ to $n$ terms
$= (a + a^2 + a^3 + \ldots$ to $n$ terms$)$
$+ b(1 + 2 + 3 + \ldots$ to $n$ terms$)$

$$= \frac{a(1-a^n)}{1-a} + b\cdot\frac{n(n+1)}{2}$$

4. Given, $\dfrac{2}{x+y}, \dfrac{1}{y}, \dfrac{2}{y+z}$ are in A.P.

$$\therefore \frac{2}{y} = \frac{2}{x+y} + \frac{2}{y+z} \text{ or } \frac{1}{y} = \frac{x+2y+z}{(x+y)(y+z)}$$

or $xy + 2y^2 + yz = xy + y^2 + xz + yz$
or $y^2 = xz$ $\therefore$ $x, y, z$ in G.P.

5. Here $a = 1$, $r = \dfrac{1}{3}$

So, $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{3}} = \dfrac{3}{2}$

6. Let $2n$ be the number of arithmetic means between two numbers $a$ and $b$.

Now sum of the $2n$ A.M.'s between $a$ and $b$

$$= \frac{a+b}{2}\cdot 2n = (a+b)n$$

Given, $(a+b)n = 2n+1 \Rightarrow \dfrac{13}{6}\cdot n = 2n+1$

$\therefore$ $n = 6$. Hence, number of means $= 2n = 12$

7. Given $x, y, z$ are in A.P.
$\therefore$ $2y = x + z$ ...(1)

$A_1$ is the A.M. of $x$ and $y$ $\therefore$ $A_1 = \dfrac{x+y}{2}$ ...(2)

$A_2$ is the A.M. of $y$ and $z$ $\therefore$ $A_2 = \dfrac{y+z}{2}$ ...(3)

Adding (2) and (3), we get

$$A_1 + A_2 = \frac{x+z+2y}{2} = \frac{2y+2y}{2} = 2y \quad \text{[From (1)]}$$

$\therefore$ $y = \dfrac{A_1 + A_2}{2}$

$\therefore$ $y$ is the A.M. of $A_1$ and $A_2$.

8. Let the numbers be $(a - d), a, (a + d)$. Then,
$(a - d) + a + (a + d) = -3 \Rightarrow a = -1$
Also, $(a - d)(a)(a + d) = 8 \Rightarrow a(a^2 - d^2) = 8$
$\Rightarrow$ $(-1)(1 - d^2) = 8$ $[\because a = -1]$
$\Rightarrow$ $d^2 = 9 \Rightarrow d = \pm 3$
When $a = -1$ and $d = 3$, the numbers are $-4, -1, 2$.
When $a = -1$ and $d = -3$, the numbers are $2, -1, -4$.
So, the numbers are $-4, -1, 2$ or $2, -1, -4$

9. Here, $a = 1$, $r = \dfrac{1}{2}$ and $n = 10$ $\therefore$ $S_{10} = a\left(\dfrac{r^{10}-1}{r-1}\right)$

$$\Rightarrow S_{10} = 1\left\{\frac{\left(\frac{1}{2}\right)^{10} - 1}{\left(\frac{1}{2}\right) - 1}\right\} = 2\left(1 - \frac{1}{2^{10}}\right)$$

$$= 2\left(\frac{2^{10}-1}{2^{10}}\right) = \frac{(1024-1)}{512} = \frac{1023}{512}$$

10. Since A.M. $\geq$ G.M.

$$\therefore \frac{x+1/x}{2} \geq \sqrt{x.\frac{1}{x}} \Rightarrow \frac{x+1/x}{2} \geq 1 \Rightarrow x + \frac{1}{x} \geq 2$$

11. $a^2 + 2bc, b^2 + 2ac, c^2 + 2ab$ are in A.P.
$\Rightarrow$ $(a^2+2bc) - (ab+bc+ca), (b^2+2ac) - (ab+bc+ca),$
$(c^2 + 2ab) - (ab + bc + ca)$ are in A.P.
$\Rightarrow$ $a^2 + bc - ab - ca, b^2 + ca - ab - bc, c^2 + ab - bc - ca$ are in A.P.
$\Rightarrow$ $(a - b)(a - c), (b - c)(b - a), (c - a)(c - b)$ are in A.P.

$$\Rightarrow \frac{(a-b)(a-c)}{(a-b)(b-c)(c-a)}, \frac{(b-c)(b-a)}{(a-b)(b-c)(c-a)},$$

$$\frac{(c-a)(c-b)}{(a-b)(b-c)(c-a)} \text{ are in A.P.}$$

$$\Rightarrow \frac{-1}{b-c}, \frac{-1}{c-a}, \frac{-1}{a-b} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b} \text{ are in A.P.}$$

**12.** Let $r$ be the common ratio of the given G.P.
Then, $b = ar$, $c = ar^2$ and $d = ar^3$
Now, L.H.S. $= (b-c)^2 + (c-a)^2 + (d-b)^2$
$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$
$= a^2r^2(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$
$= a^2[r^2(1-r)^2 + (r^2-1)^2 + r^2(r^2-1)^2]$
$= a^2[r^2(1+r^2-2r) + (r^4-2r^2+1) + r^2(r^4-2r^2+1)]$
$= a^2[r^2 + r^4 - 2r^3 + r^4 - 2r^2 + 1 + r^6 - 2r^4 + r^2]$
$= a^2[r^6 - 2r^3 + 1] = a^2(r^3-1)^2 = [a(r^3-1)]^2$
$= (ar^3 - a)^2 = (d-a)^2 = (a-d)^2$

**13.** Given, $x + y + z = 15$ ...(1)
when $a, x, y, z, b$ are in A.P.

$$\therefore \text{ Sum of A.M.'s} = x + y + z = \left(\frac{a+b}{2}\right)\cdot 3$$

$$\Rightarrow 15 = \left(\frac{a+b}{2}\right)\cdot 3 \text{ or } a+b=10 \qquad ...(2)$$

Also, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$ when $\frac{1}{a},\frac{1}{x},\frac{1}{y},\frac{1}{z},\frac{1}{b}$ are in A.P.

$$\therefore \text{ Sum of A.M.'s} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{2}\cdot 3$$

$$\text{or } \frac{5}{3} = \left(\frac{a+b}{2ab}\right)\cdot 3 = \left(\frac{10}{2ab}\right)\cdot 3 \qquad \text{[From (2)]}$$

$\therefore$ $ab = 9$
$\Rightarrow$ G.M. of $a$ and $b = \sqrt{ab} = 3$

**14.** Given $S_n = 5n^2 + 3n$
$\therefore$ $S_{n-1} = 5(n-1)^2 + 3(n-1)$
Now, $t_n = S_n - S_{n-1}$, $n \geq 2$
$= 5n^2 + 3n - 5(n-1)^2 - 3(n-1)$
$= 5[n^2 - (n-1)^2] + 3[n - n + 1]$
$= 5(2n-1) + 3 = 10n - 2$, $n \geq 2$
$\therefore$ $t_1 = S_1 = 5 \cdot 1^2 + 3\cdot 1 = 8$
$t_2 = 10 \cdot 2 - 2 = 18$
$t_3 = 10 \cdot 3 - 2 = 28$ and so on.
Thus terms of the given series are 8, 18, 28 ..... which are in A.P. whose c.d. is 10.

**15.** For general term we will take the $r^{\text{th}}$ term.
Since $r^{\text{th}}$ term of the sequence 1, 2, 3, .....
$= 1 + (r-1)\cdot 1 = r$
and $r^{\text{th}}$ term of the sequence $n, n-1, n-2$, .....
$= n + (r-1)(-1) = n - r + 1$
Now, $r^{\text{th}}$ term $(t_r) = r(n-r+1) = nr - r^2 + r$

$$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}(nr - r^2 + r)$$

$$= n\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\left(n - \frac{2n+1}{3} + 1\right)$$

$$= \frac{n(n+1)}{2}\left[\frac{3n-2n-1+3}{3}\right] = \frac{n(n+1)(n+2)}{6}$$

**16.** Given equations are $x^2 - 3x + p = 0$ ...(1)
and $x^2 - 12x + q = 0$ ...(2)
Given $a, b, c, d$ are in G.P.
Let $r$ be its common ratio.
Then, $b = ar$, $c = ar^2$, $d = ar^3$
Now $a$ and $b$ *i.e.*, $a$ and $ar$ are roots of equation (1)
$\therefore$ $a(1+r) = 3$ ...(3)
and $a^2r = p$ ...(4)
Again, $c$ and $d$ *i.e.*, $ar^2$ and $ar^3$ are the roots of equation (2)
$\therefore$ $ar^2(1+r) = 12$ ...(5)
and $a^2r^5 = q$ ...(6)
Dividing (5) by (3), we get $r^2 = 4$ $\therefore$ $r = \pm 2$

$$\text{Now } \frac{q+p}{q-p} = \frac{a^2r^5 + a^2r}{a^2r^5 - a^2r} = \frac{r^4+1}{r^4-1} = \frac{17}{15}$$

**17.** Given $a, b, c$ are in A.P. $\therefore$ $b = \frac{a+c}{2}$ ...(1)

and $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in A.P.

$$\therefore \quad \frac{2}{b^2} = \frac{1}{a^2} + \frac{1}{c^2} = \frac{a^2+c^2}{a^2c^2} \text{ or } b^2 = \frac{2a^2c^2}{a^2+c^2} \qquad ...(2)$$

$$\text{From (1) and (2), we get } \left(\frac{a+c}{2}\right)^2 = \frac{2a^2c^2}{a^2+c^2}$$

$\Rightarrow$ $(a+c)^2(a^2+c^2) = 8a^2c^2$
$\Rightarrow$ $(a^2+c^2+2ac)(a^2+c^2) = 8a^2c^2$
$\Rightarrow$ $(a^2+c^2)^2 + 2ac(a^2+c^2) - 8a^2c^2 = 0$
$\Rightarrow$ $\{(a^2+c^2)^2 - 4a^2c^2\} + \{2ac(a^2+c^2) - 4a^2c^2\} = 0$
$\Rightarrow$ $(a^2-c^2)^2 + 2ac(a^2+c^2-2ac) = 0$
$\Rightarrow$ $(a^2-c^2)^2 + 2ac(a-c)^2 = 0$
$\Rightarrow$ $(a-c)^2[(a+c)^2 + 2ac] = 0$

Either $(a-c)^2 = 0 \Rightarrow a = c$

and from (1), $b = \frac{a+a}{2} = a \Rightarrow a = b = c$

Or, $(a+c)^2 + 2ac = 0 \Rightarrow a^2 + c^2 + 4ac = 0$

or $a^2 + c^2 = -4ac$ or $\frac{2a^2c^2}{b^2} = -4ac$ [from (2)]

or $ac = -2b^2$ or $b^2 = -\frac{a}{2}\cdot c$

Hence $-\frac{a}{2}, b, c$ are in G.P.

**18.** Let $a$ be the $r^{th}$ mean between $x$ and $2y$ and $b$ be the $r^{th}$ mean between $2x$ and $y$.
Here, $n$ arithmetic means have been inserted in both cases.
$\therefore$ Number of terms of A.P. in each case = $n + 2$
Now, $a = r^{th}$ A.M. between $x$ and $2y$
$= (r+1)^{th}$ term of A.P.
$= x + rd$, where $d$ is the common difference of A.P.

$$= x + r\left(\frac{2y-x}{n+1}\right) \quad \left[\because \text{c.d.} = \frac{\text{last term} - 1^{\text{st}} \text{ term}}{\text{no. of terms} - 1}\right]$$

Similarly, $b = 2x + r\left(\frac{y-2x}{n+1}\right)$

According to question, $a = b$

$$\therefore \quad x + r\left(\frac{2y-x}{n+1}\right) = 2x + r\left(\frac{y-2x}{n+1}\right)$$

$$\Rightarrow \quad x = \frac{r}{n+1}(2y - x - y + 2x)$$

$$\Rightarrow \quad x = \frac{r}{n+1}(y+x) \Rightarrow (n+1)x = r(y+x)$$

**19.** Since $1^{st}$ group contains one number, $2^{nd}$ group contains 2 numbers, $3^{rd}$ group contains 3 numbers and so on, therefore, $n^{th}$ group will contain $n$ numbers.
Here we observe that numbers in each group are in A.P. whose c.d. is 1. Therefore, number in $n^{th}$ group will be in A.P. having c.d. 1.
Thus, for $n^{th}$ group, $d = 1$, $n = n$
Sequence of first terms of groups is 2, 4, 7, .....
First terms of this sequence is the first term of the first group.
Second term is the first term of the second group.
Third term is the first term of the third group and so on.
$n^{th}$ term of this sequence *i.e.*, $t_n$ will be the first term of the $n^{th}$ group.

Let $S_n = 1 + 2 + 4 + 7 + \ldots + t_n$ ...(1)
$S_n = \quad 1 + 2 + 4 + \ldots + t_{n-1} + t_n$ ...(2)
Subtracting (2) from (1), we get
$0 = 1 + [1 + 2 + 3 + \ldots + \text{to } (n-1) \text{ terms}] - t_n$

or $t_n = 1 + \frac{(n-1)n}{2} = \frac{n^2 - n + 2}{2}$

Thus for the $n^{th}$ group,

$a$ (first term) $= \frac{n^2-n+2}{2}$, $d = 1$, $n = n$

$\therefore$ Sum of numbers in the $n^{th}$ group

$$= \frac{n}{2}\left\{2\left(\frac{n^2-n+2}{2}\right) + (n-1)\cdot 1\right\} = \frac{n(n^2+1)}{2}$$

**20.** The sequence of the difference between successive terms is 2, 6, 18, 54, ....
This is a G.P. with first term 2 and common ratio 3.
Let $S_n = 5 + 7 + 13 + 31 \ldots + t_n$ ...(1)
Also, $S_n = \quad 5 + 7 + 13 + \ldots + t_{n-1} + t_n$ ...(2)
Subtracting (2) from (1), we get
$0 = 5 + [2 + 6 + 18 + \ldots \text{ to } (n-1) \text{ terms}] - t_n$
or $t_n = 5 + [2 + 6 + 18 + \ldots \text{ to } (n-1) \text{ terms}]$

$$= 5 + 2\frac{(3^{n-1}-1)}{(3-1)} = 5 + (3^{n-1} - 1) = 4 + 3^{n-1}$$

Now, $S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}(4 + 3^{r-1}) = \sum_{r=1}^{n} 4 + \sum_{r=1}^{n} 3^{r-1}$

$= 4n + (1 + 3 + 3^2 + \ldots + 3^{n-1})$

$$= 4n + \frac{1(3^n-1)}{3-1} = 4n + \frac{(3^n-1)}{2} = \frac{1}{2}[3^n + 8n - 1]$$

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# MPP-4 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

## Permutations & Combinations | Binomial Theorem

**Total Marks : 80** **Time Taken : 60 Min.**

### Only One Option Correct Type

**1.** A class contains three girls and four boys. Every saturday five students go on a picnic, a different group being sent each week. During the picnic, each girl in the group is given a doll by the accompanying teacher. After all possible groups of the five have gone once, the total number of dolls that the girls have got, is

(a) 27 (b) 11
(c) 21 (d) 45

**2.** For a game in which two partners play against two other partners, six persons are available. If every possible pair must play with every other possible pair, then the total number of games played is

(a) 90 (b) 45
(c) 30 (d) 60

**3.** The expression $({}^{10}C_0)^2 - ({}^{10}C_1)^2 + ({}^{10}C_2)^2 - \ldots.. + ({}^{10}C_8)^2 - ({}^{10}C_9)^2 + ({}^{10}C_{10})^2$ equals

(a) $\lfloor\underline{10}$ (b) $({}^{10}C_5)^2$
(c) $-{}^{10}C_5$ (d) ${}^{10}C_5$

**4.** At the time of Diwali festival, from a match box, having 3 red, 2 blue, 7 green sticks, 3 are taken from the box. The number of ways at least one of them is red stick, is

(a) 136 (b) 108
(c) 27 (d) 135

**5.** If the sum of the coefficients of various terms in the expansion of $(a + b)^n$ is 4096, then greatest coefficient in the expansion is

(a) 1594 (b) 792
(c) 924 (d) 2924

**6.** In the binomial expansion of $(a - b)^n$, $n \geq 5$, the sum of $5^{th}$ and $6^{th}$ terms is zero, then $a/b$ equals

(a) $\dfrac{n-5}{6}$ (b) $\dfrac{n-4}{5}$
(c) $\dfrac{5}{n-4}$ (d) $\dfrac{6}{n-5}$

### One or More Than One Option(s) Correct Type

**7.** If the term independent of $x$ in the expansion of $(\sqrt{x} - k/x^2)^{10}$ is 405, then value of $k$ is

(a) 3 (b) – 3
(c) 9 (d) –9

8. The number of seven digit integers, with sum of the digits equal to 10 and formed by using any of the digits 1, 2 and 3 only, is
(a) 55 (b) 66
(c) 77 (d) 88

9. A man has 10 friends among whom two are married to each other. Then the number of different ways in which he can invite 5 people to a dinner party if married couple refuse to attend separately is
(a) ${}^{10}C_5 - 2$ (b) ${}^{10}C_5 - 2 \times {}^{8}C_4$
(c) $2 \times {}^{8}C_3$ (d) 112

10. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then
(a) $a_1 = 20$ (b) $a_2 = 210$
(c) $a_3 = 8085$ (d) $a_{20} = 2^2 \cdot 3^7 \cdot 7$

11. Let $S = \{1, 2, ...., n\}$. If $X$ denotes the set of all subsets of $S$ containing exactly two elements, then the value of $\sum_{A \in X} (\min A)$ is given by
(a) ${}^{n+1}C_3$ (b) $\frac{1}{6}(n^2 - 1)n$
(c) ${}^{n}C_3$ (d) $\frac{1}{6}(n-1)^n$

12. If the third term in the expansion of $(x + x^{\log_{10} x})^5$ is $10^6$, then $x$ is
(a) $10^{-1/3}$ (b) 10
(c) $10^{-5/2}$ (d) $10^2$

13. In the expansion of $(x^2 + 2x + 2)^n$, (where $n$ is a positive integer) then coefficient of
(a) $x$ is $2^n \cdot n$ (b) $x^2$ is $n^2 \cdot 2^{n-1}$
(c) $x^3$ is $2^n \cdot {}^{n+1}C_3$ (d) none of these

## Comprehension Type

A committee of 12 is to be formed from 9 women and 8 men such that atleast 5 women have to be included in the committee.

14. The number of ways of forming the committee is
(a) 6000 (b) 6060
(c) 6062 (d) 6080

15. The number of committee in which women are in majority is
(a) 1008 (b) 2410
(c) 2700 (d) 2702

## Matrix Match Type

16. Match the columns:

| | Column I | | Column II |
|---|---|---|---|
| P | The sum of the two middle coefficients of $(1 + x)^9$ is | 1 | 19 |
| Q | The coefficient of $x^4$ in the expansion of $(1 + x + x^2)^4$ is | 2 | 35 |
| R | The value of $\binom{8}{0}-\binom{8}{1}+\binom{8}{2}-\binom{8}{3}+\binom{8}{4}$ is | 3 | 252 |

| | P | Q | R |
|---|---|---|---|
| (a) | 1 | 2 | 3 |
| (b) | 1 | 3 | 2 |
| (c) | 2 | 3 | 1 |
| (d) | 3 | 1 | 2 |

## Integer Answer Type

17. If ${}^{k+5}P_{k+1} = \frac{11(k-1)}{2} \times {}^{k+3}P_k$, then value of $k$ greater than 6 is

18. If $x + \frac{1}{x} = 1$ and $p = x^{4000} + \frac{1}{x^{4000}}$ and $q$ be the digit at unit place in the number $2^{2^n} + 1$, $n \in N$ and $n > 1$, then $p + q =$

19. The number of terms which are free from radical signs in the expansion of $(y^{1/5} + x^{1/10})^{55}$ is

20. A student is allowed to select at most $n$ books from a collection of $(2n + 1)$ books. If the total number of ways in which he can select a book is 63, then the value of $n$ is

◈◈

*Keys are published in this issue. Search now!* ☺

**SELF CHECK**

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

**Check your score! If your score is**

| | | |
|---|---|---|
| > 90% | EXCELLENT WORK ! | You are well prepared to take the challenge of final exam. |
| 90-75% | GOOD WORK ! | You can score good in the final exam. |
| 74-60% | SATISFACTORY ! | You need to score more next time. |
| < 60% | NOT SATISFACTORY! | Revise thoroughly and strengthen your concepts. |

# CONCEPT BOOSTERS

Class XII

## Integral Calculus

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This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

Integration of a function is the reverse process of differentiation. If $f'(x)$ is the derivative of a function $f(x)$, then the process of finding $f(x)$ from $f'(x)$ is called integration.

### ALGEBRA OF INTEGRATION

(i) $\int (U(x) \pm V(x))dx = \int U(x)dx \pm \int V(x)dx$

(ii) $\int Kf(x)\cdot dx = K\int f(x)dx$,

where $K$ is a real number (constant).

### SOME FUNDAMENTAL INTEGRALS

(i) $\int x^n \cdot dx = \dfrac{x^{n+1}}{n+1} + c$, where $n \neq -1$

(ii) $\int \dfrac{1}{x} dx = \log_e |x| + c$, where $x \neq 0$

(iii) $\int e^x \, dx = e^x + c$

(iv) $\int a^x \, dx = \dfrac{a^x}{\log_e a} + c$, where $a > 0$

(v) $\int \sin x \, dx = -\cos x + c$

(vi) $\int \cos x \, dx = \sin x + c$

(vii) $\int \sec^2 x \, dx = \tan x + c$

(viii) $\int \text{cosec}^2 x \, dx = -\cot x + c$

(ix) $\int \sec x \cdot \tan x \, dx = \sec x + c$

(x) $\int \text{cosec}\, x \cdot \cot x \, dx = -\text{cosec}\, x + c$

(xi) $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ or $-\cos^{-1} x + c$, where $|x| < 1$

(xii) $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + c$ or $-\cot^{-1} x + c$

(xiii) $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$ or $-\text{cosec}^{-1} x + c$

### SOME STANDARD INTEGRALS USING THE ABOVE RELATIONS ARE SHOWN BELOW

(i) $\int \tan x \, dx = \log|\sec x| + c = -\log|\cos x| + c$

(ii) $\int \cot x \, dx = -\log|\text{cosec}\, x| + c = \log|\sin x| + c$

(iii) $\int \sec x \, dx = \log|\sec x + \tan x| + c$

(iv) $\int \text{cosec}\, x \, dx = \log|\text{cosec}\, x - \cot x| + c$

$= \log\left|\tan \dfrac{x}{2}\right| + c$

### SOME SPECIAL INTEGRALS

(i) $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$

(ii) $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + c$

(iii) $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + c$

(iv) $\int \dfrac{dx}{\sqrt{a^2 \pm x^2}} = \log\left|x + \sqrt{x^2 \pm a^2}\right| + c$

(v) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + c, |x| < |a|$

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## INTEGRATION BY SUBSTITUTION

When integrand is a function of *i.e.*, $\int f[\phi(x)]\phi'(x)\,dx$:

Here, we put $\phi(x) = t$, so that $\phi'(x)dx = dt$ and in that case the integrand is reduced to $\int f(t)\,dt$. In this method, the integrand is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

When integrand is the product of two factors such that one is the derivative of the other *i.e.* $I = \int f'(x)\cdot f(x)\cdot dx$.

In this case we put $f(x) = t$ and convert it into a standard integral.

## EVALUATION OF THE VARIOUS FORMS OF INTEGRALS BY USE OF STANDARD RESULTS

(i) $\int \frac{lx+m}{ax^2+bx+c}dx$, where $l \neq 0$, $a \neq 0$

Take $lx+m = \frac{1}{2a}(2ax+b)+\left(m-\frac{lb}{2a}\right)$

(ii) $\int \frac{lx+m}{\sqrt{ax^2+bx+c}}dx$, where $l \neq 0$, $a \neq 0$

Put $z^2 = ax^2 + bx + c$,

so that $2z\cdot\frac{dz}{dx} = 2ax+b$ or $dz = \frac{(2ax+b)}{2z}dx$

and follow the same process as explained in (i) above.

(iii) $\int \frac{dx}{(lx+m)\sqrt{ax+b}}$, put $ax+b = z^2$

(iv) $\int \frac{dx}{(lx+m)\sqrt{ax^2+bx+c}}$, put $lx+m = \frac{1}{z}$

(v) $\int \frac{dx}{(lx^2+m)\sqrt{ax^2+b}}$, put $\sqrt{ax^2+b} = xz$ or $x = \frac{1}{z}$

(vi) $\int \frac{dx}{(x-a)^m(x-b)^n}$, where $m+n=2$

put $x - a = z(x - b)$

(vii) $\int \frac{dx}{(\text{linear})\sqrt{\text{quadratic}}}$, $\frac{1}{\text{linear}} = z$ or put as some other variable.

(viii) $\int \frac{dx}{\text{quadratic}\sqrt{\text{linear}}}$, put $\sqrt{\text{linear}} = z^2$,

or some other variable, and then follow the similar process of integration.

(ix) $\int \frac{dx}{ax^2+bx+c\,\sqrt{lx^2+mx+n}}$, put $z^2 = \frac{lx^2+mx+n}{ax^2+bx+c}$,

## INTEGRATION BY PARTS

$$\int (\text{I}\cdot\text{II})\,dx = \text{I}\int \text{II}\,dx - \int\left[\frac{d}{dx}(\text{I})\int \text{II}\,dx\right]dx + c$$

Choice of I[st] function and II[nd] function depends on order of letters in the word ILATE

I $\rightarrow$ Inverse function

L $\rightarrow$ Logarithmic function

A $\rightarrow$ Algebraic function

T $\rightarrow$ Trigonometric function

E $\rightarrow$ Exponential function

**Note :** A special integral

$$\int e^x[f(x)+f'(x)]\,dx = f(x)\cdot e^x + c$$

## PARTIAL FRACTIONS AND THEIR USES IN INTEGRATION

If the integrand is a rational function, *i.e.*, of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomial functions, depending on the nature of $p(x)$ and $q(x)$ integration can be done by the following processes:

(i) If degree $(p(x))$ < degree $(q(x))$ *i.e.*, $f(x) = \frac{mx+n}{(x-a)(x-b)}$, $a \neq b$ then we write $\frac{mx+n}{(x-a)(x-b)} = \frac{A}{x-a}+\frac{B}{x-b}$, $A$ and $B$ being constants.

(ii) If degree $(p(x))$ = degree $(q(x))$ or degree $(p(x))$ > degree $(q(x))$ of non-repeated linear factors, *i.e.*, $f(x) = \frac{mx^2+nx+l}{(x-a)(x-b)}$, $a \neq b$

then we write $\frac{mx^2+nx+l}{(x-a)(x-b)} = 1+\frac{A}{x-a}+\frac{B}{x-b}$

(iii) If denominator $q(x)$ contains linear factors, some of which are repeated, *i.e.*, integrand is of the form $\frac{p(x)}{(x-a)(x-b)^2}$, then we write the integrand as $\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{(x-b)^2}$

**Note :** To evaluate integral of the type

(i) $\int \frac{x^2+A}{x^4+kx^2+A^2}dx$

Divide numerator and denominator by $x^2$ and substitute $x - \frac{A}{x} = u$, $A$ being any positive constant.

(ii) $\int \frac{x^2 - A}{x^4 + kx^2 + A^2}\, dx$

Divide numerator and denominator by $x^2$ and substitute $x + \frac{A}{x} = t$, $A$ being positive constant.

(iii) $\int \frac{ax^2 + bx + c}{px^2 + qx + r}\, dx$

put $ax^2 + bx + c = l(px^2 + qx + r) + m\left(\frac{d}{dx}(px^2 + qx + r)\right) + n$

Find $l$, $m$ and $n$ by equating coefficients of like powers of $x$ and then split the integral into three integrals.

## TRIGONOMETRIC INTEGRALS

To find the integral $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\, dx$

Put $a$ sin$x$ + $b$ cos$x$

= $L$ (Denominator) +$M$ (Derivative of denominator)

**Note :** To evaluate the integration of the forms

$$\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}, \int \frac{dx}{a + b\sin^2 x}, \int \frac{dx}{a + b\cos^2 x},$$

$$\int \frac{dx}{(a\sin x + b\cos x)^2} \text{ and } \int \frac{dx}{a + b\sin^2 x + c\cos^2 x}$$

**Step 1 :** Divide by $\cos^2 x$ in each case.

**Step 2 :** Put $\tan x = t$ to get the form $\int \frac{dt}{at^2 + bt + c}$ which is already discussed.

## INTEGRALS OF THE FORM

(i) $\int \frac{dx}{a\sin x + b\cos x}$ (ii) $\int \frac{dx}{a + b\sin x}$

(iii) $\int \frac{dx}{a + b\cos x}$ (iv) $\int \frac{dx}{a\sin x + b\cos x + c}$

For all the cases (i), (ii), (iii) and (iv), universal substitution $\tan\frac{x}{2} = t, \sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$ &

$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$ are used. This substitution convert the integrals in the form $\int \frac{dt}{at^2 + bt + c}$.

In (i), (ii) and (iii); if $a = b$, then they becomes

$$\frac{1}{a}\int \frac{dx}{\sin x + \cos x}, \frac{1}{a}\int \frac{dx}{1 + \sin x}, \frac{1}{a}\int \frac{dx}{1 + \cos x}$$

## FUNDAMENTAL THEOREM ON CALCULUS

I. Let $f$ be a continuous function on closed interval $[\alpha, \beta]$ and $A(x)$ be the area of function. Then $A'(x) = f(x)\ \forall\ x \in [\alpha, \beta]$

II. Let $f$ be the continuous function on closed interval $[\alpha, \beta]$ and $F$ be an anti-derivative of $f$.

Then $\int_\alpha^\beta f(x)\, dx = [F(x)]_\alpha^\beta = F(\beta) - F(\alpha)$

## DEFINITE INTEGRAL AS THE LIMIT OF A SUM

An alternative method of finding $\int_\alpha^\beta f(x)\, dx$ is that the definite integral $\int_\alpha^\beta f(x)\, dx$ is a limiting case of the summation of an infinite series provided $f(x)$ is continuous on $[\alpha, \beta]$, *i.e.,*

$$\int_\alpha^\beta f(x)\, dx = \lim_{h \to 0} h[f(\alpha) + f(\alpha + h) + \ldots + f(\alpha + (n-1)h)]$$

where $h = \frac{\beta - \alpha}{n}$

## PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_\alpha^\beta f(x)\, dx = \int_\alpha^\beta f(t)\, dt$

(ii) $\int_\alpha^\beta f(x)\, dx = -\int_\beta^\alpha f(x)\, dx$

(iii) (a) $\int_\alpha^\beta f(x)\, dx = \int_\alpha^\gamma f(x)\, dx + \int_\gamma^\beta f(x)\, dx$, where $\alpha < \gamma < \beta$

(b) $\int_\alpha^\beta f(x)\, dx = \int_\alpha^{c_1} f(x)\, dx + \int_{c_1}^{c_2} f(x)\, dx + \ldots + \int_{c_n}^\beta f(x)\, dx$

(iv) $\int_0^\alpha f(x)\, dx = \int_0^\alpha f(\alpha - x)\, dx$

(v) $\int_\alpha^\beta f(x)\, dx = \int_\alpha^\beta f(\alpha + \beta - x)\, dx$

(vi) $\int_{-\alpha}^\alpha f(x)\, dx = \begin{cases} 2\int_0^\alpha f(x)dx, & \text{if } f(-x) = f(x) \\ 0 \quad , & \text{if } f(-x) = -f(x) \end{cases}$

(vii) $\int_0^{2\alpha} f(x)\, dx = \begin{cases} 2\int_0^\alpha f(x)dx, & \text{if } f(2\alpha - x) = f(x) \\ 0 \quad , & \text{if } f(2\alpha - x) = -f(x) \end{cases}$

(viii) If $f(t)$ is an odd function then $g(x) = \int_a^x f(t)dt$ is an even function.

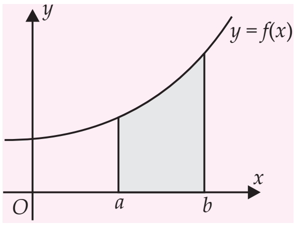
(ix) If $f(t)$ is an even function then $g(x) = \int_0^x f(t)\, dt$ is an odd function.

## LEIBNITZ'S RULE

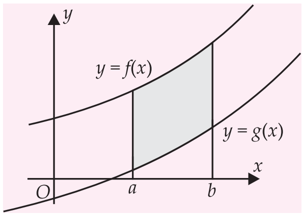
- If $f(x)$ is continuous and $u(x)$, $v(x)$ are differentiable functions in the interval $[a, b]$, then,
$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f\{v(x)\}\frac{d}{dx}\{v(x)\} - f\{u(x)\}\frac{d}{dx}\{u(x)\}$$
- If the function $\phi(x)$ and $\Psi(x)$ are defined on $[a,b]$ and differentiable at a point $x \in (a, b)$ and $f(x, t)$ is continuous, then,
$$\frac{d}{dx}\left[\int_{\phi(x)}^{\Psi(x)} f(x,t)dt\right] = \int_{\phi(x)}^{\Psi(x)} \frac{d}{dx} f(x,t)dt + \left\{\frac{d\Psi(x)}{dx}\right\} \times f(x, \Psi(x)) - \frac{d\phi(x)}{dx} f(x, \phi(x))$$

## APPLICATION OF INTEGRALS

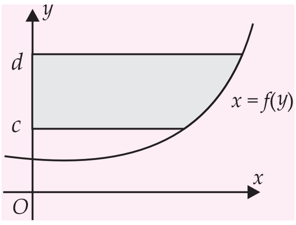
(i) The area bounded by the curve $y = f(x)$, the $x$-axis and the lines $x = a$ and $x = b$ is
$$A = \int_a^b |f(x)|dx$$



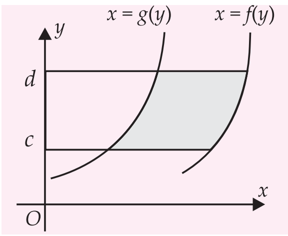

(ii) The area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is
$$A = \int_a^b |f(x) - g(x)|dx$$




(iii) The area bounded by the curve $x = f(y)$, the $y$-axis and the lines $y = c$ and $y = d$ is
$$A = \int_c^d |f(y)|dy$$




(iv) The area bounded by the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ is
$$A = \int_c^d |f(y) - g(y)|dy$$




## PROBLEMS

### Single Correct Answer Type

**1.** $\int \frac{\sin x}{\sin(x-\alpha)}dx =$

(a) $x\cos\alpha - \sin\alpha \log\sin(x - \alpha) + c$
(b) $x\cos\alpha + \sin\alpha \log\sin(x - \alpha) + c$
(c) $x\sin\alpha - \sin\alpha \log\sin(x - \alpha) + c$
(d) None of these

**2.** $\int \frac{\cos 2x - 1}{\cos 2x + 1}dx =$

(a) $\tan x - x + c$ (b) $x + \tan x + c$
(c) $x - \tan x + c$ (d) $-x - \cot x + c$

**3.** $\int \frac{(x+1)^2}{x(x^2+1)}dx$ is equal to

(a) $\log_e x + c$ (b) $\log_e x + 2\tan^{-1}x + c$
(c) $\log_e\left(\frac{1}{x^2+1}\right) + c$ (d) $\log_e\{x(x^2 + 1)\} + c$

**4.** $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x}dx =$

(a) $\log(x^e + e^x) + c$ (b) $e\log(x^e + e^x) + c$
(c) $\frac{1}{e}\log(x^e + e^x) + c$ (d) None of these

**5.** $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx =$

(a) $\cot^{-1}(\tan^2 x) + c$ (b) $\tan^{-1}(\tan^2 x) + c$
(c) $\cot^{-1}(\cot^2 x) + c$ (d) $\tan^{-1}(\cot^2 x) + c$

**6.** $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}dx =$

(a) $\frac{1}{b^2}\log(a^2 + b^2\sin^2 x) + c$
(b) $\frac{1}{b}\log(a^2 + b^2\sin^2 x) + c$
(c) $\log(a^2 + b^2\sin^2 x) + c$
(d) $b^2\log(a^2 + b^2\sin^2 x) + c$

**7.** $\int \frac{\cos 2x}{(\cos x + \sin x)^2}dx =$

(a) $\log\sqrt{\cos x + \sin x} + c$ (b) $\log(\cos x - \sin x) + c$
(c) $\log(\cos x + \sin x) + c$ (d) $-\frac{1}{\cos x + \sin x} + c$

**8.** $\int \frac{1}{\sqrt{1 - e^{2x}}}dx =$

(a) $x - \log[1 + \sqrt{1 - e^{2x}}] + c$
(b) $x + \log[1 + \sqrt{1 - e^{2x}}] + c$
(c) $\log[1 + \sqrt{1 - e^{2x}}] - x + c$ (d) None of these

**9.** $\int \frac{\sin 2x}{\sin 5x \sin 3x}dx =$

(a) $\log\sin 3x - \log\sin 5x + c$

(b) $\frac{1}{3}\log\sin 3x + \frac{1}{5}\log\sin 5x + c$

(c) $\frac{1}{3}\log\sin 3x - \frac{1}{5}\log\sin 5x + c$

(d) $3\log\sin 3x - 5\log\sin 5x + c$

**10.** $\int \frac{dx}{x\log x \cdot \log(\log x)} =$

(a) $2\log(\log x) + c$ (b) $\log[\log(\log x)] + c$

(c) $\log(x\log x) + c$ (d) None of these

**11.** $\int \frac{1+x^2}{\sqrt{1-x^2}}dx =$

(a) $\frac{3}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c$

(b) $\frac{3}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + c$

(c) $\frac{3}{2}\cos^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c$

(d) $\frac{3}{2}\cos^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + c$

**12.** $\int \sqrt{\frac{x}{a^3 - x^3}}dx =$

(a) $\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$ (b) $\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$

(c) $\frac{3}{2}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$ (d) $\frac{3}{2}\sin^{-1}\left(\frac{x}{a}\right)^{2/3} + c$

**13.** $\int \frac{x^5 dx}{\sqrt{(1+x^3)}} =$

(a) $\frac{2}{3}\sqrt{(1+x^3)}(x^3+2) + c$

(b) $\frac{2}{9}\sqrt{(1+x^3)}(x^3-4) + c$

(c) $\frac{2}{9}\sqrt{(1+x^3)}(x^3+4) + c$

(d) $\frac{2}{9}\sqrt{(1+x^3)}(x^3-2) + c$

**14.** $\int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}dx$ is equal to

(a) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$ (b) $\frac{4}{3}\left(\frac{x+1}{x-2}\right)^{1/4} + c$

(c) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$ (d) $\frac{1}{3}\left(\frac{x+1}{x-1}\right)^{1/4} + c$

**15.** $\int \frac{1}{1+\sin^2 x}dx =$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + k$

(b) $\sqrt{2}\tan^{-1}(\sqrt{2}\tan x) + k$

(c) $-\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + k$

(d) $-\sqrt{2}\tan^{-1}(\sqrt{2}\tan x) + k$

**16.** $\int \frac{x^2\tan^{-1}(x^3)}{1+x^6}dx$ is equal to

(a) $\tan^{-1}(x^3) + c$ (b) $\frac{1}{6}(\tan^{-1}(x^3))^2 + c$

(c) $-\frac{1}{2}(\tan^{-1}(x^3))^2 + c$ (d) $\frac{1}{2}(\tan^{-1}(x^2))^3 + c$

**17.** $\int \frac{\sin^3 2x}{\cos^5 2x}dx =$

(a) $\tan^4 x + c$ (b) $\tan 4x + c$

(c) $\tan^4 2x + x + c$ (d) $\frac{1}{8}\tan^4 2x + c$

**18.** The value of $\int \frac{2dx}{\sqrt{1-4x^2}}$ is

(a) $\tan^{-1}(2x) + c$ (b) $\cot^{-1}(2x) + c$

(c) $\cos^{-1}(2x) + c$ (d) $\sin^{-1}(2x) + c$

**19.** If $\int f(x)dx = g(x)$, then $\int f^{-1}(x)dx$ is equal to

(a) $g^{-1}(x)$ (b) $xf^{-1}(x) - g(f^{-1}(x))$

(c) $xf^{-1}(x) - g^{-1}(x)$ (d) $f^{-1}(x)$

**20.** $\int \frac{\sin x}{\sin x - \cos x}dx =$

(a) $\frac{1}{2}\log(\sin x - \cos x) + x + c$

(b) $\frac{1}{2}[\log(\sin x - \cos x) + x] + c$

(c) $\frac{1}{2}\log(\cos x - \sin x) + x + c$

(d) $\frac{1}{2}[\log(\cos x - \sin x) + x] + c$

21. $\int \left\{\frac{(\log x - 1)}{1+(\log x)^2}\right\}^2 dx$ is equal to

(a) $\frac{xe^x}{1+x^2}+c$ (b) $\frac{x}{(\log x)^2+1}+c$

(c) $\frac{\log x}{(\log x)^2+1}+c$ (d) $\frac{x}{x^2+1}+c$

22. Let $f(x)=\int \frac{x^2 dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0)=0$, then the value of $f(1)$ be

(a) $\log(1+\sqrt{2})$ (b) $\log(1+\sqrt{2})-\frac{\pi}{4}$

(c) $\log(1+\sqrt{2})+\frac{\pi}{4}$ (d) none of these

23. $\int \cos^{-3/7} x \cdot \sin^{-11/7} x dx =$

(a) $\log|\sin^{4/7} x|+c$ (b) $\frac{4}{7}\tan^{4/7} x+c$

(c) $\frac{-7}{4}\tan^{-4/7} x+c$ (d) $\log|\cos^{3/7}x| + c$

24. $\int_0^{\pi/2} \sqrt{\cos\theta}\sin^3\theta d\theta =$

(a) $\frac{20}{21}$ (b) $\frac{8}{21}$ (c) $\frac{-20}{21}$ (d) $\frac{-8}{21}$

25. $\int_a^b \frac{\log x}{x} dx =$

(a) $\log\left(\frac{\log b}{\log a}\right)$ (b) $\log(ab)\log\left(\frac{b}{a}\right)$

(c) $\frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$ (d) $\frac{1}{2}\log(ab)\log\left(\frac{a}{b}\right)$

26. $\int_0^1 \tan^{-1} x dx =$

(a) $\frac{\pi}{4}-\frac{1}{2}\log 2$ (b) $\pi-\frac{1}{2}\log 2$

(c) $\frac{\pi}{4}-\log 2$ (d) $\pi-\log 2$

27. $\int_0^{\pi/2} \frac{dx}{2+\cos x} =$

(a) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\sqrt{3}\tan^{-1}(\sqrt{3})$

(c) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $2\sqrt{3}\tan^{-1}(\sqrt{3})$

28. $\int_0^a \frac{x^4 dx}{(a^2+x^2)^4} =$

(a) $\frac{1}{16a^3}\left(\frac{\pi}{4}-\frac{1}{3}\right)$ (b) $\frac{1}{16a^3}\left(\frac{\pi}{4}+\frac{1}{3}\right)$

(c) $\frac{1}{16}a^3\left(\frac{\pi}{4}-\frac{1}{3}\right)$ (d) $\frac{1}{16}a^3\left(\frac{\pi}{4}+\frac{1}{3}\right)$

29. $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x} dx =$

(a) $\frac{1}{20}\log 3$ (b) log3

(c) $\frac{1}{20}\log 5$ (d) none of these

30. $\int_0^{\pi/2} \frac{\sin x\cos x dx}{\cos^2 x+3\cos x+2} =$

(a) $\log\left(\frac{8}{9}\right)$ (b) $\log\left(\frac{9}{8}\right)$

(c) $\log(8\times 9)$ (d) none of these

31. The value of the integral

$\int_{-\pi}^{\pi} \sin mx \sin nx dx$ for $m \neq n (m, n \in I)$, is

(a) 0 (b) $\pi$ (c) $\frac{\pi}{2}$ (d) $2\pi$

32. $\int_0^{\pi} \frac{dx}{1-2a\cos x+a^2} =$

(a) $\frac{\pi}{2(1-a^2)}$ (b) $\pi(1 - a^2)$

(c) $\frac{\pi}{1-a^2}$ (d) none of these

33. The value of $\int \frac{dx}{1+e\cos x}$ must be same as

(a) $\frac{1}{\sqrt{1-e^2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{x}{2}\right)+c$

($e$ lies between 0 and 1)

(b) $\frac{2}{\sqrt{1-e^2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{x}{2}\right)+c$,

($e$ lies between 0 and 1)

(c) $\dfrac{1}{\sqrt{e^2-1}}\log\dfrac{\sqrt{e^2-1}\sin x}{1+e\cos x}+c,$ ($e$ is greater than 1)

(d) $\dfrac{2}{\sqrt{e^2-1}}\log\dfrac{e+\cos x+\sqrt{e^2-1}\sin x}{1+e\cos x}+c,$ ($e$ is greater than 1)

**34.** Let $f$ be a differentiable function such that $f'(x)=f(x)+\int_0^2 f(x)dx,\ f(0)=\dfrac{4-e^2}{3}$, then $f(x)$ is

(a) $e^x-\left(\dfrac{e^2-1}{3}\right)$ (b) $e^x-\dfrac{(e^2-2)}{3}$

(c) $e^x+\dfrac{e-[\arg|z-1|]}{3}$ (d) None of these

## Multiple Correct Answer Type

**35.** If $I=\int_0^{\frac{\pi}{2}}\dfrac{\sin(\sin x)}{\sin x}dx,\ I_2=\int_0^{\frac{\pi}{2}}\dfrac{\sin x}{x}dx$ and $I_3=\int_0^{\frac{\pi}{2}}\dfrac{\sin(\tan x)}{\tan x}dx$, then which of the following is true?

(a) $I_1>I_3$ (b) $I_2>I_3$

(c) $I_1>I_2$ (d) $I_1<I_2$

**36.** If $f(x)$ is monotonic and differentiable function, then $\int_{f(a)}^{f(b)} 2x(b-f^{-1}(x))dx=$

(a) $\int_a^b (f^2(x)-f^2(a))dx$ (b) $\int_a^b (f^2(x)-f^2(b))dx$

(c) $\int_a^b f^2(x)dx+(a-b)f^2(b)$

(d) $\int_a^b f^2(x)dx+(a-b)f^2(a)$

**37.** $\int\dfrac{dx}{(1+\sqrt{x})^{2010}}=2\left[\dfrac{1}{\alpha(1+\sqrt{x})^{\alpha}}-\dfrac{1}{\beta(1+\sqrt{x})^{\beta}}\right]+c$ where $\alpha,\beta>0$ then

(a) $|\alpha-\beta|=1$

(b) $(\beta+2)(\alpha+1)=(2010)^2$

(c) $\beta,\alpha$, 2010 are in A.P.

(d) $\alpha+1=\beta+2=2010$

**38.** $\int_{-1/2}^{1/2}\sqrt{\left\{\left(\dfrac{x+1}{x-1}\right)^2+\left(\dfrac{x-1}{x+1}\right)^2-2\right\}}dx$ is

(a) $4\ln\left(\dfrac{4}{3}\right)$ (b) $4\ln\left(\dfrac{3}{4}\right)$

(c) $-\ln\left(\dfrac{81}{256}\right)$ (d) $\ln\left(\dfrac{256}{81}\right)$

**39.** Let $f(x)=\int_{\pi^2/4}^{x^2}\dfrac{\sin x}{1+\cos^2\sqrt{t}}dt$ then

(a) $f'\left(\dfrac{\pi}{2}\right)=\pi$ (b) $f'\left(-\dfrac{\pi}{2}\right)=\pi$

(c) $f'\left(\dfrac{3\pi}{2}\right)=-3\pi$ (d) $f'(\pi)=\int_{\pi^2}^{\pi^2/4}\dfrac{dx}{1+\cos^2\sqrt{x}}$

**40.** The value of $\int_{1/e}^{\tan x}\dfrac{tdt}{1+t^2}+\int_{1/e}^{\cot x}\dfrac{dt}{t(1+t^2)}$ is

(a) $\dfrac{1}{2+\tan^2 x}$ (b) 1

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{2}{\pi}\int_{-1}^{1}\dfrac{dt}{1+t^2}$

## Comprehension Type

### Paragraph for Q. No. 41 to 43

Let $f(x)$ defined in $[a, b]$ has discontinuities $C_1, C_2, C_3$, ...., $C_n$ such that $a<C_1<C_2<....<C_n<b$ then

$$\int_a^b f(x)dx=\int_a^{C_1}f(x)dx+\int_{C_1}^{C_2}f(x)dx+....+\int_{C_{n-1}}^{C_n}f(x)dx+\int_{C_n}^{b}f(x)dx$$

**41.** $\int_{-1}^{1}[2x-3]dx=$ (where [.] is greatest integer function)

(a) $-7$ (b) $-9$ (c) 5 (d) 11/2

**42.** $\int_0^{50\pi}[\tan^{-1}x]dx=$ (where [.] is greatest integer function)

(a) $\tan 1+50\pi$ (b) $-\tan 1$

(c) $50\pi-\tan 1$ (d) $20\pi-2\tan 1$

43. $\int_{\pi/2}^{3\pi/2} [\sin x]dx =$ (where [.] is greatest integer function)

(a) $-\pi$ (b) $\pi/2$ (c) $-\pi/2$ (d) $\pi$

## Matrix-Match Type

44. Match the following :

| | Column I | | Column II |
|---|---|---|---|
| A. | The value of $\int_{\alpha}^{\pi/2-\alpha} \frac{dx}{1+\cot^n x}$ where, $0<\alpha<\frac{\pi}{2}, n>0$ is | P. | $\frac{\pi}{2}$ |
| B. | The value of $\int_{-\pi}^{\pi} \frac{\sin^2 x}{1+\alpha^x} dx, \alpha>0$ is | Q. | $\frac{\pi}{4}-\alpha$ |
| C. | The value of $\int_{\alpha}^{\frac{3\pi}{2}-\alpha} \frac{\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx$ is | R. | $\frac{3\pi}{4}-2\alpha$ |
| D. | The value of $\int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha} \frac{\tan x}{\tan x+\cot x} dx$ is | S. | $\frac{\pi}{4}-\tan^{-1}\alpha$ |

45. Match the following :

| | Column I | | Column II |
|---|---|---|---|
| A. | $\int \frac{\sec x}{(\sec x+\tan x)^2} dx =$ | P. | $\log\left\|\frac{\sin x-2}{\sin x-1}\right\|+C$ |
| B. | $\int \frac{\cos x}{(\sin x-1)(\sin x-2)} dx =$ | Q. | $-\frac{\cos^2 x}{2(1+\sin x)^2}+C$ |
| C. | $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx,$ $\|x\|<1=$ | R. | $2x\tan^{-1} x - \log(1+x^2) + C$ |
| D. | $\int (\sqrt{\tan x}+\sqrt{\cot x})dx =$ | S. | $\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+C$ |

## Integer Answer Type

46. If the value of definite integral $\int_1^a x\cdot a^{-[\log_a x]}dx$ where $a>1$, and [.]denotes the greatest integer, is $\frac{e-1}{2}$ then the value of $5[a]$ is ____

47. If $I=\int_0^{\pi} x(\sin^2(\sin x)+\cos^2(\cos x))dx$, then $[I]$ = _____, where [.] denotes the greatest integer function

48. Area bounded by $2 \ge \max.\{|x-y|, |x+y|\}$ is $k$ sq. units then $k =$

49. The area bounded by the curves $y = \ln x$, $y = \ln|x|$, $y = |\ln x|$, $y = |\ln|x||$(in sq. units) is

50. Let $f(x) = x^3 + 3x + 2$ and $g(x)$ is the inverse of it. The area bounded by $g(x)$, the $x$-axis and the ordinates at $x = -2$ and $x = 6$ is $\frac{m}{n}$ where $m, n \in N$ and G.C.D of $(m, n) = 1$ then $m - 2 =$

51. The integral $\int_{\pi/4}^{5\pi/4} (|\cos t|\sin t+|\sin t|\cos t)dt$ has the value equal to

52. If the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the line $y = 3x - 15$ is $\frac{73}{\lambda}$ sq. units, then the value of $\lambda$ is

53. The minimum area bounded by the function $y = f(x)$ and $y = \alpha x + 9$, $(\alpha \in R)$ where $f$ satisfies the relation $f(x + y) = f(x) + f(y) + y\sqrt{f(x)}$ $\forall x, y \in R$ and $f'(0) = 0$ is $9A$, then value of $A$ is

54. Let $R = \{x, y : x^2 + y^2 \le 144 \text{ and } \sin(x + y) \ge 0\}$ and $S$ be the area of region given by $R$, then find $S/9\pi$.

55. If the area bounded by $[x] + [y] = n$ and $y = k$; $n, k \in N$ and $k \le (n+1)$ and [.] greatest integer function, in the first quadrant, is $n + r$, then find $r$.

## SOLUTIONS

1. (b) : $\int \frac{\sin x}{\sin(x-\alpha)}dx = \int \frac{\sin(x-\alpha+\alpha)}{\sin(x-\alpha)}dx$

$= \int \frac{(\sin(x-\alpha)\cos\alpha+\cos(x-\alpha)\sin\alpha}{\sin(x-\alpha)}dx$

$= \int \cos\alpha dx + \int \sin\alpha\cdot\cot(x-\alpha)dx$

$= x\cos\alpha + \sin\alpha \cdot \log\sin(x - \alpha) + c$

**2. (c) :** Let $I = \int \frac{\cos 2x - 1}{\cos 2x + 1} dx$

$\Rightarrow \quad I = -\int \frac{(1-\cos 2x)}{(1+\cos 2x)} dx = -\int \frac{2\sin^2 x}{2\cos^2 x} dx$

$\Rightarrow \quad I = -\int \tan^2 x dx = -\int (\sec^2 x - 1) dx$

$\Rightarrow \quad I = x - \tan x + c$

**3. (b) :** $\int \frac{(x+1)^2}{x(x^2+1)} dx = \int \frac{x^2+1+2x}{x(x^2+1)} dx$

$= \int \frac{x^2+1}{x(x^2+1)} dx + 2\int \frac{x}{x(x^2+1)} dx$

$= \int \frac{dx}{x} + 2\int \frac{dx}{x^2+1} = \log_e x + 2\tan^{-1} x + c$

**4. (c) :** Put $x^e + e^x = t \Rightarrow e(x^{e-1} + e^{x-1})dx = dt$

Now, $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} dx = \frac{1}{e}\int \frac{dt}{t} = \frac{1}{e}\log t = \frac{1}{e}\log(x^e + e^x) + c$

**5. (b) :** Let $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

$= \int \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{2\tan x \sec^2 x}{1+\tan^4 x} dx$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x dx = dt$

$\therefore \quad I = \int \frac{dt}{1+t^2} = \tan^{-1} t + c = \tan^{-1}(\tan^2 x) + c$

**6. (a) :** Put $a^2 + b^2 \sin^2 x = t \Rightarrow b^2 \sin 2x dx = dt$, then

$\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx = \frac{1}{b^2}\int \frac{dt}{t} = \frac{1}{b^2}\log t + c$

$= \frac{1}{b^2}\log(a^2 + b^2 \sin^2 x) + c$

**7. (c) :** $\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$

$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Put $t = \sin x + \cos x \Rightarrow dt = (\cos x - \sin x)dx$, then it reduces to

$\int \frac{1}{t} dt = \log t + c = \log(\sin x + \cos x) + c$

**8. (a) :** $\int \frac{1}{\sqrt{1-e^{2x}}} dx = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}} dx$

Put $e^{-x} = t \Rightarrow -e^{-x} dx = dt$, then it reduces to

$-\int \frac{1}{\sqrt{t^2-1}} dt = -\log[t + \sqrt{t^2-1}] + c$

$= -\log[e^{-x} + \sqrt{e^{-2x}-1}] = -\log\left[\frac{1}{e^x} + \frac{\sqrt{1-e^{2x}}}{e^x}\right]$

$= -\log[1 + \sqrt{1-e^{2x}}] + \log e^x + c = x - \log[1 + \sqrt{1-e^{2x}}] + c$

**9. (c) :** $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx = \int \frac{\sin(5x-3x)}{\sin 5x \sin 3x} dx$

$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x} dx$

$= \frac{1}{3}\log \sin 3x - \frac{1}{5}\log \sin 5x + c$

**10. (b) :** $\int \frac{dx}{x \log x \cdot \log(\log x)}$

Put $\log(\log x) = z$

$\Rightarrow \quad \frac{1}{\log x} \cdot \frac{1}{x} dx = dz$, then it reduces to

$\int \frac{dz}{z} = \log z = \log[\log(\log x)] + c$

**11. (a) :** Put $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$, then

$\int \frac{1+x^2}{\sqrt{1-x^2}} dx = \int (1+\sin^2\theta) d\theta = \theta + \frac{1}{2}\int (1-\cos 2\theta) d\theta$

$= \frac{3\theta}{2} - \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} + c = \frac{3}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + c$

**12. (b) :** Put $x = a(\sin\theta)^{2/3}$

$\Rightarrow \quad dx = \frac{2}{3}a(\sin\theta)^{-1/3}\cos\theta d\theta$

$\therefore \quad \int \sqrt{\frac{x}{a^3 - x^3}} dx = \int \frac{a^{1/2}(\sin\theta)^{1/3} \frac{2}{3}a(\sin\theta)^{-1/3}\cos\theta}{\sqrt{a^3 - a^3\sin^2\theta}} d\theta$

$= \frac{2}{3}a^{3/2}\int \frac{\cos\theta d\theta}{a^{3/2}\sqrt{1-\sin^2\theta}} = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$

**13. (d) :** Put $1 + x^3 = t^2 \Rightarrow 3x^2 dx = 2t dt$

$\therefore \quad I = \frac{2}{3}\int (t^2 - 1) dt$

$= \frac{2}{3}\left(\frac{t^3}{3} - t\right) + c = \frac{2}{9}t(t^2-3) + c = \frac{2}{9}(x^3-2)\sqrt{1+x^3} + c$

**14. (a) :** Let $I = \int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}} dx$

$= \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2} dx$

Put $\frac{x-1}{x+2} = t \Rightarrow \frac{3}{(x+2)^2} dx = dt$

$\therefore \quad I=\frac{1}{3}\int\frac{1}{t^{3/4}}dt=\frac{1}{3}\left(\frac{t^{1/4}}{1/4}\right)+c=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+c$

**15. (a) :** $I=\int\frac{1}{1+\sin^2 x}dx=\int\frac{dx}{2\sin^2 x+\cos^2 x}$

$=\int\frac{\sec^2 x dx}{2\tan^2 x+1}=\frac{1}{2}\int\frac{\sec^2 x dx}{\tan^2 x+\frac{1}{2}}$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, then

$I=\frac{1}{2}\int\frac{dt}{t^2+\frac{1}{2}}=\frac{1}{2}\left(\frac{1}{1/\sqrt{2}}\right)\tan^{-1}\left(\frac{t}{1/\sqrt{2}}\right)+k$

$=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+k$

**16. (b) :** Put $\tan^{-1}(x^3)=z \Rightarrow \frac{1}{1+x^6}\times 3x^2dx=dz$

Now, $\int\frac{x^2\tan^{-1}(x^3)}{1+x^6}dx=\frac{1}{3}\int z dz$

$=\frac{1}{3}\cdot\frac{z^2}{2}=\frac{1}{6}(\tan^{-1}(x^3))^2+c$

**17. (d) :** $I=\int\frac{\sin^3 2x}{\cos^5 2x}dx=\int\tan^3 2x\cdot\sec^2 2x dx$

Putting $\tan 2x = t$ and $2\sec^2 2x dx = dt$, we get

$I=\int\frac{t^3 dt}{2}=\frac{1}{2}\cdot\frac{t^4}{4}+c=\frac{1}{8}(\tan^4 2x)+c$

**18. (d) :** Put $2x=\sin\theta \Rightarrow 2dx=\cos\theta d\theta$

$\Rightarrow \quad I=\int\frac{\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta=\int d\theta=\theta+c$

$\Rightarrow \quad I=\sin^{-1}(2x)+c$

**19. (b) :** $\int f(x)dx=g(x)$ (Given)

Now, $I=\int f^{-1}(x)\cdot dx=f^{-1}(x)\int dx-\int\left\{\frac{d}{dx}f^{-1}(x)\int dx\right\}dx$

$=xf^{-1}(x)-\int\left\{x\frac{d}{dx}f^{-1}(x)\right\}dx=xf^{-1}(x)-\int x d\{f^{-1}(x)\}$

Let $f^{-1}(x)=t \Rightarrow x=f(t)$ and $d\{f^{-1}(x)\}=dt$

$\therefore I=xf^{-1}(x)-\int f(t)dt=xf^{-1}(x)-g(t)=xf^{-1}(x)-g\{f^{-1}(x)\}$

**20. (b) :** $\int\frac{\sin x}{\sin x-\cos x}dx=\frac{1}{2}\int\frac{2\sin x}{\sin x-\cos x}dx$

$=\frac{1}{2}\int\frac{(\sin x-\cos x+\sin x+\cos x)}{\sin x-\cos x}dx$

$=\frac{1}{2}\int\left(1+\frac{\sin x+\cos x}{\sin x-\cos x}\right)dx=\frac{1}{2}[x+\log(\sin x-\cos x)]+c$

**21. (b) :** Put $\log x=t \Rightarrow dx=e^t dt$

$\int\left\{\frac{\log x-1}{1+(\log x)^2}\right\}^2 dx=\int e^t\left[\frac{1}{1+t^2}-\frac{2t}{(1+t^2)^2}\right]dt$

$=\frac{e^t}{1+t^2}+c=\frac{x}{1+(\log x)^2}+c$

**22. (b) :** Let $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

$f(x)=\int\frac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}=\int\frac{\tan^2\theta\sec^2\theta d\theta}{\sec^2\theta(1+\sec\theta)}$

$=\int\frac{\tan^2\theta d\theta}{1+\sec\theta}=\int\frac{\sin^2\theta d\theta}{\cos\theta(1+\cos\theta)}=\int\frac{(1-\cos^2\theta)d\theta}{\cos\theta(1+\cos\theta)}$

$=\int\frac{(1-\cos\theta)d\theta}{\cos\theta}=\int\sec\theta d\theta-\int d\theta$

$=\log(x+\sqrt{1+x^2})-\tan^{-1}x+c$

Now, $f(0)=\log(0+\sqrt{1+0})-\tan^{-1}(0)+c \Rightarrow c=0$

$\therefore \quad f(1)=\log(1+\sqrt{1+1^2})-\tan^{-1}(1)=\log(1+\sqrt{2})-\frac{\pi}{4}$

**23. (c) :** $I=\int\cos^{-3/7}x\cdot(\sin^{(-2+3/7)}x)dx$

$=\int\cos^{-3/7}x\sin^{-2}x\sin^{3/7}x\,dx$

$=\int\frac{\operatorname{cosec}^2 x}{\left(\frac{\cos^{3/7}x}{\sin^{3/7}x}\right)}dx=\int\frac{\operatorname{cosec}^2 x dx}{\cot^{3/7}x}$

Put $\cot x=t \Rightarrow -\operatorname{cosec}^2 x dx=dt$

$I=-\int\frac{dt}{t^{3/7}}=-\frac{7}{4}t^{4/7}+c=-\frac{7}{4}\tan^{-4/7}x+c$

**24. (b) :** Let $I=\int_0^{\pi/2}\sqrt{\cos\theta}\sin^3\theta d\theta$

Put $t=\cos\theta \Rightarrow dt=-\sin\theta d\theta$

$I=-\int_1^0 t^{1/2}(1-t^2)dt=\int_0^1(t^{1/2}-t^{5/2})dt$

$\Rightarrow \quad I=\left[\frac{2}{3}t^{3/2}-\frac{2}{7}t^{7/2}\right]_0^1=\frac{8}{21}$

**25. (c) :** Let $I=\int_a^b\frac{1}{x}\log x dx$

$\Rightarrow \quad I=[\log x\cdot\log x]_a^b-\int_a^b\frac{1}{x}\log x dx$

$\Rightarrow \quad 2I=[(\log x)^2]_a^b \Rightarrow I=\frac{1}{2}[(\log b)^2-(\log a)^2]$

$\Rightarrow \quad I=\frac{1}{2}[(\log b+\log a)(\log b-\log a)]=\frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$

**26. (a) :** Put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

Also as $x = 0, \theta = 0$ and $x = 1, \theta = \frac{\pi}{4}$

Therefore, $\int_0^1 \tan^{-1} x dx = \int_0^{\pi/4} \theta \sec^2\theta d\theta$

$$= \frac{\pi}{4} - \log\sqrt{2} = \frac{\pi}{4} - \frac{1}{2}\log 2$$

**27. (c) :** $I = \int_0^{\pi/2} \frac{dx}{2+\cos x}$

$$= \int_0^{\pi/2} \frac{dx}{2\sin^2\frac{x}{2} + 2\cos^2\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \int_0^{\pi/2} \frac{dx}{\sin^2\frac{x}{2} + 3\cos^2\frac{x}{2}} = \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{3+\tan^2\frac{x}{2}} dx$$

Put $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx$, then

$$I = 2\int_0^1 \frac{dt}{3+t^2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

**28. (a) :** Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta d\theta$

$$\therefore \quad I = \int_0^{\pi/4} \frac{a^4\tan^4\theta \cdot a\sec^2\theta}{a^8\sec^8\theta} d\theta$$

$$= \frac{1}{a^3}\int_0^{\pi/4} \sin^4\theta\cos^2\theta d\theta = \frac{1}{a^3}\left[\int_0^{\pi/4} (\sin^4\theta - \sin^6\theta) d\theta\right]$$

$$= \frac{1}{a^3}\int_0^{\pi/4}\left[\frac{(1-\cos 2\theta)^2}{4} - \frac{(1-\cos 2\theta)^3}{8}\right] d\theta$$

$$= \frac{1}{8a^3}\int_0^{\pi/4} (1+\cos 2\theta)(1+\cos^2 2\theta - 2\cos 2\theta) d\theta$$

$$= \frac{1}{8a^3}\int_0^{\pi/4} (1-\cos 2\theta - \cos^2 2\theta + \cos^3 2\theta) d\theta$$

$$= \frac{1}{32a^3}\int_0^{\pi/4} (2-\cos 2\theta - 2\cos 4\theta + \cos 6\theta) d\theta$$

$$= \frac{1}{32a^3}\left[2\theta - \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 6\theta}{6}\right]_0^{\pi/4} = \frac{1}{16a^3}\left(\frac{\pi}{4} - \frac{1}{3}\right)$$

**29. (a) :** Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx$

Put $\sin x - \cos x = t \Rightarrow (\sin x + \cos x)dx = dt$

$$I = \int_{-1}^{0} \frac{dt}{9+16(1-t^2)} = \int_{-1}^{0} \frac{dt}{25-16t^2}$$

$$I = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2} = \frac{1}{16}\left[\frac{2}{5}\log\left|\frac{5+4x}{5-4x}\right|\right]_{-1}^{0}$$

$$= \frac{1}{40}[\log 1 - \log 1 + \log 9] = \frac{1}{20}\log 3$$

**30. (b) :** Let $I = \int_0^{\pi/2} \frac{\sin x\cos x dx}{\cos^2 x + 3\cos x + 2}$

Put $\cos x = t \Rightarrow -\sin x dx = dt$, then

$$I = \int_0^1 \frac{t dt}{t^2+3t+2} = \int_0^1 \left[\frac{2}{t+2} - \frac{1}{t+1}\right] dt$$

$$= [2\log(t+2) - \log(t+1)]_0^1 = [2\log 3 - \log 2 - 2\log 2]$$

$$= [\log 9 - \log 8] = \log\left(\frac{9}{8}\right)$$

**31. (a) :** Let $I = 2\int_0^{\pi} \sin mx \sin nx dx$

$$= \int_0^{\pi} [\cos(m-n)x - \cos(m+n)x] dx$$

$$= \left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{(m+n)}\right]_0^{\pi}$$

$$= \left[\frac{\sin(m-n)\pi}{(m-n)} - \frac{\sin(m+n)\pi}{(m+n)}\right] = 0$$

Since, $\sin(m-n)\pi = 0 = \sin(m+n)\pi$ for $m \neq n$

**32. (c) :** $\int_0^{\pi} \frac{dx}{(1+a^2)\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) - 2a\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$

$$= \int_0^{\pi} \frac{dx}{(1-a)^2\cos^2\frac{x}{2} + (1+a)^2\sin^2\frac{x}{2}}$$

$$= \frac{2}{(1+a)^2}\int_0^{\infty} \frac{dt}{\{(1-a)/(1+a)\}^2 + t^2}; \quad \left\{\text{where } t = \tan\frac{x}{2}\right\}$$

$$= \frac{2}{(1+a)^2}\frac{(1+a)}{(1-a)}\left[\tan^{-1}\left(\frac{1+a}{1-a}\cdot t\right)\right]_0^{\infty}$$

$$= \frac{2}{(1-a^2)}[\tan^{-1}\infty - \tan^{-1}0] = \frac{\pi}{1-a^2}$$

**33. (b)**

**34. (a) :** $f'(x) = f(x) + k$

$$\int \frac{f'(x)}{f(x)+k}dx = \int dx$$

$\Rightarrow$ $\log(f(x) + k) = x + C \Rightarrow f(x) = k_1e^x - k$

$$\therefore \quad f(0) = k_1 - k = \frac{4-e^2}{3} \quad \text{.... (1)}$$

Also, $k = \int_0^2 f(x)dx \;\Rightarrow\; 3k = k_1(e^2 - 1)$ .... (2)

Solving (1) and (2), we get

$$f(x) = e^x - \left(\frac{e^2-1}{3}\right)$$

**35. (a, b, c) :** $0 < x < \frac{\pi}{2} \Rightarrow \frac{\sin x}{x}$

is decreasing and $\sin x < x < \tan x$

$$\Rightarrow \frac{\sin(\sin x)}{\sin x} > \frac{\sin x}{x} > \frac{\sin(\sin x)}{\tan x} \Rightarrow I_1 > I_2 > I_3$$

**36. (a, d) :** Let $f^{-1}(x) = y \;\Rightarrow\; x = f(y)$

$\Rightarrow$ $dx = f'(y)dy$

$$I = \int_a^b 2f(y)(b-y)f'(y)dy$$

$$= b\int_a^b 2f(y)\,f'(y)dy - \int_a^b 2yf(y)\,f'(y)dy$$

$$= b(f^2(b) - f^2(a)) - bf^2(b) + af^2(a) + \int_a^b f^2(y)\,dy$$

$$= \int_a^b f^2(x)dx + (a-b)f^2(a) = \int_a^b (f^2(x) - f^2(a))dx$$

**37. (a, b, c, d) :** $\int \frac{dx}{(1+\sqrt{x})^{2010}} = \int \frac{\sqrt{x}}{\sqrt{x}(1+\sqrt{x})^{2010}}dx$

$$= 2\int \frac{t-1}{t^{2010}}dt \quad (\text{where } t = 1+\sqrt{x})$$

$$= 2\left[\frac{1}{2009\,t^{2009}} - \frac{1}{2008\,t^{2008}}\right] + c$$

$\Rightarrow$ $\alpha = 2009, \beta = 2008$

**38. (a, c, d) :** Let $I = \int_{-1/2}^{1/2} \sqrt{\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right)^2}\,dx$

$$= \int_{-1/2}^{1/2} \left|\frac{4x}{x^2-1}\right| dx = -2\int_0^{1/2} \frac{4x}{(x^2-1)}dx$$

$$= -4\left\{\ln|x^2-1|\right\}_0^{1/2} = -4\ln\left(\frac{3}{4}\right)$$

$$= 4\ln\left(\frac{4}{3}\right) = \ln\left(\frac{256}{81}\right) = -\ln\left(\frac{81}{256}\right)$$

**39. (a, b, c, d) :**

$$f'(x) = \sin x \times \frac{2x}{1+\cos^2 x} + \cos x \int_{\pi^2/4}^{x^2} \frac{1}{1+\cos^2\sqrt{t}}dt$$

**40. (b, d) :** Let $I = \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$

Put $t = 1/z \Rightarrow dt = -\frac{1}{z^2}dz$

$$\therefore \quad I = \int_e^{\tan x} \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\left(1+\frac{1}{z^2}\right)} = \int_{\tan x}^e \frac{zdz}{(z^2+1)} = \int_{\tan x}^e \frac{tdt}{(1+t^2)}$$

$$\therefore \quad \int_{1/e}^{\tan x} \frac{tdt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$= \int_{1/e}^{\tan x} \frac{t}{1+t^2}dt + \int_{\tan x}^{e} \frac{t}{1+t^2}dt = \int_{1/e}^{e} \frac{tdt}{1+t^2} = \frac{1}{2}[\ln(1+t^2)]_{1/e}^{e}$$

$$= \frac{1}{2}\left\{\ln(1+e^2) - \ln\left(1+\frac{1}{e^2}\right)\right\} = \frac{1}{2}(\ln e^2) = 1$$

Also, $\frac{2}{\pi}\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{4}{\pi}\int_0^1 \frac{dt}{1+t^2} = \frac{4}{\pi}\cdot\tan^{-1}1 = \frac{4}{\pi}\cdot\frac{\pi}{4} = 1$

**41. (a) :** $\int_{-1}^{-1/2} -5dx + \int_{-1/2}^{0} -4dx + \int_0^{1/2} -3dx + \int_{1/2}^{1} -2dx$

$$= -\frac{5}{2} - 2 - \frac{3}{2} - 1 = -7$$

**42. (c) :** $\int_0^{\tan 1} 0\,dx + \int_{\tan 1}^{50\pi} 1\,dx = 50\pi - \tan 1$

**43. (c) :** $\int_{\pi/2}^{\pi} 0\,dx + \int_{\pi}^{3\pi/2} -1\,dx = -\frac{\pi}{2}$

**44. A → (Q), B → (P), C → (R), D → (S)**

**(A)** Let $I = \int_{\alpha}^{\pi/2-\alpha} \frac{\sin^n x}{\sin^n x + \cos^n x}dx$ ... (i)

$$I = \int_{\alpha}^{\pi/2-\alpha} \frac{\sin^n\left(\frac{\pi}{2}-\alpha\right)}{\sin^n\left(\frac{\pi}{2}-\alpha\right) + \cos^n\left(\frac{\pi}{2}-\alpha\right)}dx$$

$$= \int_{\alpha}^{\pi/2-\alpha} \frac{\cos^n\alpha}{\cos^n\alpha + \sin^n\alpha} \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I=\int_{\alpha}^{\pi/2-\alpha}1\cdot dx=\frac{\pi}{2}-2\alpha \quad\Rightarrow\quad I=\frac{\pi}{4}-\alpha$$

**(B)** Let $I=\int_{-\pi}^{\pi}\frac{\sin^2 x}{1+\alpha^x}dx$ ... (i)

$$\Rightarrow\quad I=\int_{-\pi}^{\pi}\frac{\sin^2(0-x)}{1+\alpha^{0-x}}dx \Rightarrow I=\int_{-\pi}^{\pi}\frac{\alpha^x\sin^2 x}{1+\alpha^x}dx \text{ ... (ii)}$$

Adding (i) and (ii), we get

$$2I=\int_{-\pi}^{\pi}\sin^2 xdx=2\int_0^{\pi}\sin^2 xdx$$

$$\Rightarrow I=\int_0^{\pi}\sin^2 xdx=2\int_0^{\pi/2}\sin^2 xdx=\frac{\pi}{2}$$

**(C)** Let $I=\int_{\alpha}^{\frac{3\pi}{2}-\alpha}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ ... (i)

$$=\int_{\alpha}^{\frac{3\pi}{2}-\alpha}\frac{\sin^{2n}\left(\frac{3\pi}{2}-x\right)}{\sin^{2n}\left(\frac{3\pi}{2}-x\right)+\cos^{2n}\left(\frac{3\pi}{2}-x\right)}dx$$

$$\Rightarrow\quad I=\int_{\alpha}^{\frac{3\pi}{2}-\alpha}\frac{\cos^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \text{ ... (ii)}$$

Adding (i) and (ii), we get

$$2I=\int_{\alpha}^{\frac{3\pi}{2}-\alpha}1\cdot dx=\left(\frac{3\pi}{2}-\alpha-\alpha\right) \quad\Rightarrow\quad I=\frac{3\pi}{4}-\alpha$$

**(D)** Let $I=\int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha}\frac{\tan x}{\tan x+\cot x}dx$ ... (i)

$$=\int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha}\frac{\cot x}{\cot x+\tan x}dx \text{ ... (ii)}$$

Adding (i) and (ii), we get

$$2I=\int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha}1dx=\cot^{-1}\alpha-\tan^{-1}\alpha$$

$$=\left(\frac{\pi}{2}-\tan^{-1}\alpha\right)-\tan^{-1}\alpha \text{ or } I=\left(\frac{\pi}{4}-\tan^{-1}\alpha\right)$$

**45. A → (Q), B → (P), C → (R), D → (S)**

**(A)** We have $\int\frac{\sec x}{(\sec x+\tan x)^2}dx$

$$=\int\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^3}dx=\int\frac{dt}{t^3}\text{(Putting } t=\sec x+\tan x)$$

$$=-\frac{1}{2}(\sec x+\tan x)^{-2}+C$$

**(B)** $\int\frac{\cos x}{(\sin x-1)(\sin x-2)}dx=\int\frac{dt}{(t-1)(t-2)}$

(Putting $t=\sin x$)

$$=\int\left(\frac{1}{t-2}-\frac{1}{t-1}\right)dt=\log\frac{|\sin x-2|}{|\sin x-1|}+C$$

**(C)** $\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx=2\int t\sec^2 tdt$ (Putting $x=\tan t$)

$= 2[t\tan t-\log|\sec t|+C_1]=2x\tan^{-1}x-\log(1+x^2)+C$

**(D)** Putting $\tan x=y^2$, so that $dx=\frac{2ydy}{(1+y^4)}$, we have

$$\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\left(y+\frac{1}{y}\right)\frac{2y}{1+y^4}dy$$

$$=2\int\frac{y^2+1}{1+y^4}dy=2\int\frac{1+1/y^2}{y^2+1/y^2}dy$$

$$=2\int\frac{1+1/y^2}{(y-1/y)^2+2}dy=2\int\frac{du}{u^2+2}\text{ where } u=y-\frac{1}{y}$$

$$=2\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+C=\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+C$$

**46. (5) :** Let $\log_a x=t \Rightarrow a^t=x \Rightarrow dx=a^t\log_e a\,dt$

$$\therefore\quad \int_1^a x\cdot a^{-[\log_a x]}dx=\ln a\int_0^1 a^t\cdot a^{-[t]}\cdot a^t\,dt=\ln a\int_0^1 a^{2t}dt$$

$$=\frac{a^2-1}{2}=\frac{e-1}{2} \quad\Rightarrow\quad a=\sqrt{e}$$

**47. (4) :** $I=\int_0^{\pi}(\pi-x)((\sin^2(\sin x))+\cos^2(\cos x))dx$

$$\Rightarrow\quad 2I=2\pi\int_0^{\pi/2}(\sin^2(\sin x)+\cos^2(\cos x))dx$$

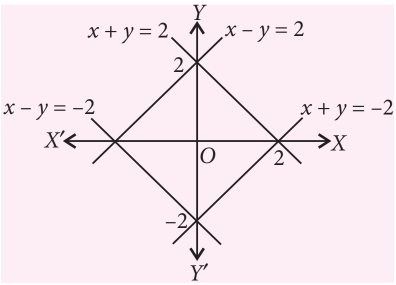
$$\Rightarrow\quad I=\pi\int_0^{\pi/2}(\sin^2(\sin x)+\cos^2(\cos x))\,dx$$

$$=\pi\int_0^{\pi/2}(\sin^2(\cos x)+\cos^2(\sin x))\,dx$$

$$\Rightarrow\quad 2I=\pi\int_0^{\pi/2}2dx \quad\Rightarrow\quad I=\frac{\pi^2}{2}$$

**48. (8) :** $2 \geq \max.\{|x-y|, |x+y|\}$

$\Rightarrow$ $|x-y| \leq 2$ and $|x+y| \leq 2$, which forms a square of diagonal length 4 units.

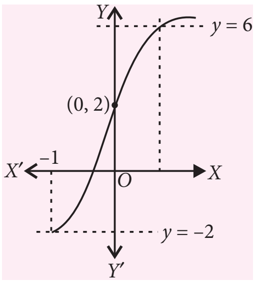



$\Rightarrow$ The area of the region $= \frac{1}{2} \times 4 \times 4 = 8$ sq. units.

**49. (4) :** Area $= 4\int_0^1 |\ln x|\,dx$

$$= -4\int_0^1 \ln x\,dx = -4[x\ln x - x]_0^1 = 4 \text{ sq. units}$$

**50. (7) :** Then required area will be equal to area enclosed by $y = f(x)$, the $y$-axis between the abscissa at $y = -2$ and $y = 6$.




Required area $= \int_0^1 \{6 - f(x)\}dx + \int_{-1}^0 [f(x) - (-2)]dx$

$$\Rightarrow \frac{9}{2} = \frac{m}{n} \Rightarrow m - 2 = 7$$

**51. (0) :** $I = \int_{\pi/4}^{\pi/2} 2\sin t\cos t\,dt$

$$+ \int_{\pi/2}^{\pi} \{(-\sin t\cos t) + (\sin t\cos t)\}dt + \int_{\pi}^{5\pi/4} -2\sin t\cos t\,dt$$

$$= \int_{\pi/4}^{\pi/2} \sin 2t\,dt - \int_{\pi}^{5\pi/4} \sin 2t\,dt = 0$$

**52. (6) :** Area $= \left|\int_1^5 (6x - x^2 - 5)dx - \int_1^3 (4x - x^2 - 3)dx\right|$

$$+ \left|\int_3^4 (4x - x^2 - 3)dx\right| + \left|\int_4^5 (3x - 15)dx\right|$$

$$= \frac{73}{6} \text{ sq. units}$$

**53. (8) :** $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x+0)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + h\sqrt{f(x)} - f(x) - f(0) - 0\sqrt{f(x)}}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)}$$

$$\Rightarrow f'(x) = \sqrt{f(x)} \Rightarrow \int \frac{f'(x)}{\sqrt{f(x)}}dx = \int dx$$

$$\Rightarrow 2\sqrt{f(x)} = x + c \Rightarrow f(x) = \frac{x^2}{4}$$

When $\alpha = 0$, area is minimum.

Required minimum area $= 2\int_0^9 2\sqrt{y}\,dy$

$$= 4\left(\frac{y^{3/2}}{3/2}\right)_0^9 = 72 \text{ sq. units.}$$

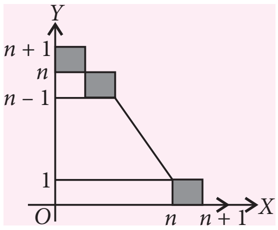
**54. (8) :** $x^2 + y^2 \leq 144$ and $\sin(x+y) \geq 0$

$\Rightarrow$ $2n\pi \leq x + y \leq (2n+1)\pi$ ; $n \in I$

Hence, we get the area

$$S = \frac{\pi \cdot 144}{2} \Rightarrow \frac{S}{9\pi} = 8$$

**55. (1) :** Area $= n + 1$



CLASS XII Series 6

# ACE YOUR WAY CBSE

## Differential Equations

### HIGHLIGHTS

**DEFINITION**

An equation involving the derivative(s) of dependent variable $y$ w.r.t. independent variable $x$ or equation of dependent variables with respect to independent variables involving derivative is called a differential equation.

A differential equation involving derivatives with respect to only one independent variable is called ordinary differential equation.

**ORDER AND DEGREE OF A DIFFERENTIAL EQUATION**

- The order of the highest order derivative occurring in the given differential equation is called the order of the differential equation.
- The power of the highest order derivative occurring in the differential equation is called the degree of the differential equation.

**Note :** Order and degree (if defined) of a differential equation are always positive integers.

**SOLUTION OF A DIFFERENTIAL EQUATION**

A relation between the independent and dependent variables free from derivatives satisfying the given differential equation is called a solution of the given differential equation.

**GENERAL AND PARTICULAR SOLUTIONS OF A DIFFERENTIAL EQUATION**

- The general solution of a differential equation is the relation between the variables (not involving the differential coefficients) satisfying the given differential equation and containing as many arbitrary constants as its order is.

| Previous Years Analysis | | | | | | |
|---|---|---|---|---|---|---|
| | 2016 | | 2015 | | 2014 | |
| | Delhi | AI | Delhi | AI | Delhi | AI |
| VSA | – | – | 2 | 2 | - | - |
| SA | 2 | 2 | - | - | 3 | 3 |
| LA | - | - | 1 | 1 | - | - |

- The solution of the differential equation for particular values of one or more of the arbitrary constants is called a particular solution of the given differential equation.

**FORMATION OF A DIFFERENTIAL EQUATION WHOSE GENERAL SOLUTION IS GIVEN**

Suppose an equation of a family of curves contains $n$ arbitrary constants (called parameters).

Then, we obtain its differential equation, using following steps :

**Step I :** Differentiate the equation of the given family of curves $n$ times to get $n$ more equations.

**Step II :** Eliminate $n$ constants, using these $(n + 1)$ equations.

This gives us the required differential equation of order $n$.

**METHODS OF SOLVING FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS**

**(i)** If the equation is $\frac{dy}{dx} = f(x)$, then $y = \int f(x)dx + C$ is the solution.

**(ii)** **Variable separable :** If the given differential equation can be expressed in the form $f(x)dx = g(y)dy$, then $\int f(x)dx = \int g(y)dy + C$ is the solution.

is

**(iii) Reducible to variable separable :** If the equation $\frac{dy}{dx} = f(ax + by + c)$, then put $ax + by + c = z$.

**(iv) Homogeneous equation :** If a first order, first degree differential equation is expressible in the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree in $x$ and $y$, then put $y = vx$.

**(v) Linear equation :** If the equation is $\frac{dy}{dx} + Py = Q$, where $P$ and $Q$ are functions of $x$, then $y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + C$, where $e^{\int P dx}$ is the integrating factor (I.F.).

**OR**

If the equation is $\frac{dx}{dy} + Px = Q$, where $P$ and $Q$ are functions of $y$, then

$x \cdot e^{\int P dy} = \int Q \cdot e^{\int P dy} dy + C$, where $e^{\int P dy}$ is the integrating factor (I.F.).

## PROBLEMS

### Very Short Answer Type

1. Solve the differential equation $\frac{dy}{dx} = 1 - x + y - xy$.
2. Determine the order and degree of the differential equation $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.
3. Find the differential equation of the family of all straight lines.
4. Find the integrating factor of the differential equation $\left\{\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right\}\frac{dx}{dy} = 1 (x \neq 0)$.
5. Find the order and degree of the differential equation $y = px + \sqrt{a^2p^2 + b^2}$, where $p = \frac{dy}{dx}$.

### Short Answer Type

6. Obtain the differential equation of the family of curves represented by $y = Ae^x + Be^{-x} + x^2$, where $A$ and $B$ are arbitrary constants.
7. Solve the differential equation $\log\left(\frac{dy}{dx}\right) = ax + by$.
8. Solve : $(1 + xy)ydx + (1 - xy)xdy = 0$
9. Verify that $y = ae^{3x} + be^{-x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$
10. Solve the following differential equation $\frac{dy}{dx} + \frac{y}{x} = e^x;\ x > 0$

### Long Answer Type-I

11. Solve the differential equation $(1 + e^{2x})dy + e^x(1 + y^2)dx = 0$. Given that $y = 1$, when $x = 0$.
12. Solve the following differential equation $\frac{dy}{dx} + 2y = \sin x$
13. Find the differential equation of the family of all circles touching the $x$-axis at the origin.
14. Show that the curve for which the normal at every point passes through a fixed point is a circle.
15. Solve the following differential equation $\frac{dy}{dx} + \sec x \cdot y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$

### Long Answer Type-II

16. Solve the following differential equation $(x^3 + y^3)dy - x^2ydx = 0$
17. In a bank, principal increases at the rate of 5% per year. In how many years Rs. 1000 double itself?
18. Find the general solution of the following differential equation $y\,dx - (x + 2y^2)dy = 0$.
19. Solve the following differential equation $$2xy + y^2 - 2x^2\frac{dy}{dx} = 0;\ y(1) = 2.$$
20. Solve the following differential equation $$\frac{dy}{dx} + y\cot x = x^2\cot x + 2x$$ Also find the particular solution, given that $y = 0$ when $x = \frac{\pi}{2}$.

## SOLUTIONS

1. We have $\frac{dy}{dx} = (1 - x)(1 + y) \Rightarrow \frac{dy}{1 + y} = (1 - x)dx$

$$\Rightarrow \int \frac{dy}{1 + y} = \int (1 - x)dx \Rightarrow \log|1 + y| = x - \frac{x^2}{2} + C$$

2. We have $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

On squaring, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

Clearly, order of differential equation is 2 and degree is also 2.

**3.** The general equation of the family of all straight lines is given by $y = mx + c$, where $m$ and $c$ are parameters.

Now, $y = mx + c \Rightarrow \frac{dy}{dx} = m \Rightarrow \frac{d^2y}{dx^2} = 0$

So, the required differential equation is $\frac{d^2y}{dx^2} = 0$

**4.** We have $\frac{dy}{dx} + \frac{1}{\sqrt{x}} \cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

I.F. $= e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$.

**5.** We have $y = px + \sqrt{a^2p^2 + b^2}$

$\Rightarrow y - px = \sqrt{a^2p^2 + b^2}$

$\Rightarrow (y - px)^2 = a^2p^2 + b^2$ [On squaring both sides]

$\Rightarrow y^2 + x^2p^2 - 2xyp = a^2p^2 + b^2$

$\Rightarrow (x^2 - a^2)p^2 - 2xyp + (y^2 - b^2) = 0$

$\Rightarrow (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy \cdot \left(\frac{dy}{dx}\right) + (y^2 - b^2) = 0$

Clearly, it is a differential equation of order 1 and degree 2.

**6.** We have $y = Ae^x + Be^{-x} + x^2$ ...(1)

Differentiating (1) w.r.t. $x$, we get

$\frac{dy}{dx} = Ae^x - Be^{-x} + 2x$ ...(2)

Again differentiating (2) w.r.t. $x$, we get

$\frac{d^2y}{dx^2} = Ae^x + Be^{-x} + 2$ ...(3)

$(1) - (3) \Rightarrow y - \frac{d^2y}{dx^2} = x^2 - 2$

or, $\frac{d^2y}{dx^2} - y + x^2 - 2 = 0$, which is the required differential equation.

**7.** We have $\log\left(\frac{dy}{dx}\right) = ax + by$

$\Rightarrow \frac{dy}{dx} = e^{ax+by} = e^{ax} \cdot e^{by} \Rightarrow \frac{1}{e^{by}} dy = e^{ax} dx$

$\Rightarrow \int e^{-by} dy = \int e^{ax} dx \Rightarrow \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C$

$\Rightarrow ae^{-by} + be^{ax} = K$, where $K = -abC$.

This is the required solution of the given differential equation.

**8.** Given, $(1 + xy)ydx + (1 - xy)xdy = 0$

$\Rightarrow (ydx + xdy) + xy(ydx - xdy) = 0$

$\Rightarrow d(xy) = xy(xdy - ydx)$

$\Rightarrow \frac{d(xy)}{x^2y^2} = \frac{xdy - ydx}{xy} \Rightarrow \frac{d(xy)}{(xy)^2} = \frac{dy}{y} - \frac{dx}{x}$

Integrating, we get $-\frac{1}{xy} = \log y - \log x + c$

$\Rightarrow -\frac{1}{xy} = \log\left(\frac{y}{x}\right) + c$, which is the required solution.

**9.** We have $y = ae^{3x} + be^{-x}$

$\therefore \frac{dy}{dx} = 3ae^{3x} - be^{-x}$ and $\frac{d^2y}{dx^2} = 9ae^{3x} + be^{-x}$

Now, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = (9ae^{3x} + be^{-x})$

$-2(3ae^{3x} - be^{-x}) - 3(ae^{3x} + be^{-x}) = 0$

Hence, $y = ae^{3x} + be^{-x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$

**10.** We have $\frac{dy}{dx} + \frac{1}{x}y = e^x; (x > 0)$

This is a linear D.E. of the form $\frac{dy}{dx} + Py = Q$

Where, $P = \frac{1}{x}$ and $Q = e^x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ The solution is given by

$yx = \int xe^x dx + C$

or $yx = e^x(x - 1) + C$

This is the required solution of the given differential equation.

**11.** We have $(1 + e^{2x})dy + e^x(1 + y^2)dx = 0$

$\Rightarrow \frac{dy}{1 + y^2} + \frac{e^x}{1 + e^{2x}} dx = 0$

Integrating, we get

$$\tan^{-1} y + \int \frac{dt}{1+t^2} = c \qquad \text{(Where } t = e^x\text{)}$$

$\Rightarrow \tan^{-1}y + \tan^{-1}t = c \Rightarrow \tan^{-1}y + \tan^{-1}(e^x) = c$

When $x = 0, y = 1$

$\Rightarrow \tan^{-1} 1 + \tan^{-1}(e^0) = c$

or $c = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \quad \therefore \quad \tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2}$

This is the required solution of the given differential equation.

**12.** We have $\frac{dy}{dx} + 2y = \sin x$

This is a linear D.E. of the form $\frac{dy}{dx} + Py = Q$

Where, $P = 2$ and $Q = \sin x$

Now, I.F. $= e^{\int P dx} = e^{\int 2dx} = e^{2x}$

$\therefore$ The solution is given by

$y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$

$\Rightarrow ye^{2x} = \int e^{2x} \sin x\, dx + c$

$= \frac{e^{2x}}{5}(2\sin x - \cos x) + c$

$\Rightarrow y = \frac{1}{5}(2\sin x - \cos x) + ce^{-2x}$

This is required solution of the given differential equation.

**13.** Equation of a circle with centre $(0, a)$ and radius $a$ is

$x^2 + (y - a)^2 = a^2$ or $x^2 + y^2 = 2ay$ ...(1)

where $a$ is a parameter.

Differentiating both sides of (1) w.r.t. $x$, we get

$$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx} \quad \text{or} \quad a\frac{dy}{dx} = x + y\frac{dy}{dx}$$

or $a = \frac{x}{\left(\frac{dy}{dx}\right)} + y$ ...(2)

Putting the value of $a$ from (2) in (1), we get

$$x^2 + y^2 = 2y\left\{\frac{x}{\left(\frac{dy}{dx}\right)} + y\right\} \quad \text{or} \quad (x^2 - y^2) = \frac{2xy}{\left(\frac{dy}{dx}\right)}$$

or $(x^2 - y^2)\frac{dy}{dx} - 2xy = 0$

This is the required differential equation.

**14.** Let $P(x, y)$ be an arbitrary point on the given curve. The equation of the normal to the given curve at $(x, y)$ is $Y - y = -\frac{1}{\frac{dy}{dx}}(X - x)$

It is given that the normal at every point passes through a fixed point $(\alpha, \beta)$ (say).

Therefore, $\beta - y = -\frac{dx}{dy}(\alpha - x)$

$\Rightarrow (x - \alpha)dx + (y - \beta)dy = 0$

Integrating both sides, we get

$\Rightarrow \int (x - \alpha)dx + \int (y - \beta)dy = C$

$\Rightarrow \frac{(x-\alpha)^2}{2} + \frac{(y-\beta)^2}{2} = C$

$\Rightarrow (x - \alpha)^2 + (y - \beta)^2 = r^2$, where $r^2 = 2C$

Clearly, this equation represents a circle, having centre at $(\alpha, \beta)$ and radius $r$.

**15.** We have $\frac{dy}{dx} + \sec x \cdot y = \tan x$ ...(1)

This is a linear D.E. of the form $\frac{dy}{dx} + Py = Q$.

Where, $P = \sec x$ and $Q = \tan x$

Now, I.F. $= e^{\int P dx} = e^{\int \sec x dx}$

$= e^{\log(\sec x + \tan x)} = \sec x + \tan x$

$\therefore$ The solution of (1) is given by

$y \cdot (\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + c$

$= \int (\sec x \tan x + \tan^2 x)dx + c$

$= \int \sec x \tan x\, dx + \int \sec^2 x\, dx - \int 1 \cdot dx + c$

$\therefore y(\sec x + \tan x) = \sec x + \tan x - x + c$

This is the required solution of the given differential equation.

**16.** We have $(x^3 + y^3)dy - x^2y dx = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$ ...(1)

This is a homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(2)

Now, differential equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{v}{1+v^3} \Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v = -\frac{v^4}{1+v^3}$$

$\Rightarrow \frac{1+v^3}{v^4}dv + \frac{dx}{x} = 0$

Integrating both sides, we get

$$\int \left[\frac{1}{v^4} + \frac{1}{v}\right]dv + \int \frac{dx}{x} = 0$$

$\Rightarrow -\frac{1}{3v^3} + \log|v| + \log|x| = c$

$$\Rightarrow -\frac{1}{3v^3}+\log|vx|=c \Rightarrow -\frac{x^3}{3y^3}+\log|y|=c$$

This is the required solution of the given differential equation.

**17.** Let $P$ be the principal at any time $t$. Then,

$$\frac{dP}{dt}=5\% \text{ of } P=\frac{5P}{100}$$

$$\Rightarrow \frac{dP}{dt}=\frac{P}{20} \Rightarrow \frac{1}{P}dP=\frac{1}{20}dt$$

Integrating both sides, we get

$$\int\frac{1}{P}dP=\int\frac{1}{20}dt$$

$$\Rightarrow \log P=\frac{1}{20}t+\log C \Rightarrow \log\frac{P}{C}=\frac{1}{20}t$$

$$\Rightarrow P=Ce^{t/20} \qquad ...(1)$$

When $t = 0$, we have $P = 1000$

Putting these values in (1), we get $1000 = C$

$$\Rightarrow P = 1000\, e^{t/20} \qquad ...(2)$$

Let $T$ years be the time required to double the principal *i.e.*, at $t = T$, $P = 2000$. Substituting these values in (2), we get

$$2000 = 1000e^{T/20}$$

$$\Rightarrow e^{T/20}=2 \Rightarrow \frac{T}{20}=\log_e 2 \Rightarrow T=20\log_e 2$$

Hence, the principal doubles in $20 \log_e 2$ years.

**18.** We have $y\,dx -(x + 2y^2)dy = 0$

$$\text{or } \frac{dx}{dy}=\frac{x+2y^2}{y} \text{ or } \frac{dx}{dy}+x\left(-\frac{1}{y}\right)=2y \qquad ...(1)$$

This differential equation is of the form

$$\frac{dx}{dy}+Px=Q, \text{ where } P=-\frac{1}{y} \text{ and } Q=2y$$

$$\text{Now, I.F.}=e^{\int Pdy}=e^{-\int\frac{1}{y}dy}=e^{-\log y}=\frac{1}{y}$$

$\therefore$ Solution is given by $x\cdot\frac{1}{y}=\int Q\cdot\frac{1}{y}dy+C$

$$\Rightarrow \frac{x}{y}=\int 2y\cdot\frac{1}{y}dy+C=2y+C \Rightarrow x = 2y^2 + Cy$$

This is the general solution of the given differential equation.

**19.** We have $2xy+y^2-2x^2\frac{dy}{dx}=0 \qquad ...(1)$

and $y(1) = 2$

$$\text{From (1), } \frac{dy}{dx}=\frac{2xy+y^2}{2x^2} \qquad ...(2)$$

This is a linear homogeneous differential equation.

Put $y = vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ Equation (2) becomes

$$v+x\frac{dv}{dx}=\frac{2v+v^2}{2}=v+\frac{1}{2}v^2$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1}{2}v^2 \Rightarrow -2\frac{dv}{v^2}+\frac{dx}{x}=0$$

Integrating, we get

$$\frac{2}{v}+\log|x|=c \Rightarrow \frac{2x}{y}+\log|x|=c \qquad ...(3)$$

Since $y(1) = 2$

$$\therefore \text{ From (3), } \frac{2\times1}{2}+\log 1=c \Rightarrow c=1$$

$$\Rightarrow \frac{2x}{y}+\log|x|=1 \Rightarrow 2x - y + y\log|x| = 0$$

This is the required particular solution.

**20.** We have $\frac{dy}{dx}+y\cot x=x^2\cot x+2x \qquad ...(1)$

This is a linear D.E. of the form $\frac{dy}{dx}+Py=Q$

Where $P = \cot x$ and $Q = x^2\cot x + 2x$

$$\therefore \text{ I.F.}=e^{\int Pdx}=e^{\int\cot x\,dx}$$

$$=e^{\log \sin x} = \sin x$$

$\therefore$ The solution of (1) is given by

$$y\cdot\sin x=\int(x^2\cot x+2x)\sin x dx+c$$

$$=\int x^2\cos x dx+2\int x\sin x dx+c$$

$$=x^2\sin x-\int 2x\cdot\sin x dx+2\int x\sin x dx+c$$

$$= x^2\sin x + c$$

$$\Rightarrow y = x^2 + c\cdot \operatorname{cosec} x \qquad ...(2)$$

Since, $y=0$, when $x=\frac{\pi}{2}$

$$\therefore \text{ From (2), } 0=\frac{\pi^2}{4}+c \Rightarrow c=-\frac{\pi^2}{4}$$

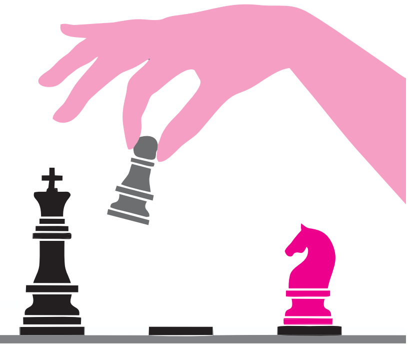
$$\Rightarrow y=x^2-\frac{\pi^2}{4}\operatorname{cosec}x$$

This is the required particular solution of the given differential equation.

## MPP-4 CLASS XI ANSWER KEY

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | (d) | **2.** | (b) | **3.** | (c) | **4.** | (a) | **5.** | (c) |
| **6.** | (b) | **7.** | (a, b) | **8.** | (c) | **9.** | (b,c,d) | **10.** | (a,b,c) |
| **11.** | (a,b) | **12.** | (b,c) | **13.** | (a,b,c) | **14.** | (c) | **15.** | (d) |
| **16.** | (d) | **17.** | (7) | **18.** | (6) | **19.** | (6) | **20.** | (3) |

# Challenging PROBLEMS ON CALCULUS



## SECTION-I

### Single Correct Answer Type

**1.** A continuous function $f:[0,1]\to R$ satisfies the equation $\int_0^1 f(x)dx=\frac{1}{3}+\int_0^1 f^2(x^2)\,dx$, then $f\left(\frac{1}{4}\right)=$

(a) 1 (b) 1/2 (c) 1/4 (d) –1/4

**2.** A continuous function $f:[0,1]\to R$ satisfies the condition $\int_0^1 f(x)(x-f(x))dx=\frac{1}{12}$, then $f\left(\frac{1}{4}\right)=$

(a) 1/2 (b) 1/4 (c) 1/6 (d) 1/8

**3.** Compute $\int_0^1 (2x^3-3x^2-x+1)^{1/3}dx$

(a) 0 (b) –1 (c) 1/8 (d) –1/8

**4.** Consider a one-to-one differentiable solution $y$ to the equation $\frac{d^2y}{dx^2}+\frac{d^2x}{dy^2}=0$ such that $y(0)=2$ and $y(-1)=-1$ then $y(1)=$

(a) –20 (b) 20 (c) –5 (d) 5

**5.** Let $f:[0,1]\to R$ be a continuous function with the property that $x\,f(y)+y\,f(x)\le 1$ for all $x, y\in[0,1]$ then

(a) $\int_0^1 f(x)dx=\frac{\pi}{4}$ (b) $\int_0^1 f(x)dx\le\frac{\pi}{4}$

(c) $\int_0^1 f(x)dx\ge\frac{\pi}{4}$ (d) $\int_0^1 f(x)dx\ge\frac{\pi}{2}$

**6.** For every $\theta\in(0,\pi]$, we have

(a) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt>\sqrt{\theta^2+\sin^2\theta}$

(b) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt=\sqrt{\theta^2+\sin^2\theta}$

(c) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt<\sqrt{\theta^2+\sin^2\theta}$

(d) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt=\frac{1}{2}\sqrt{\theta^2+\sin^2\theta}$

**7.** $y(x)$ satisfies the differential equation $\frac{dy}{dx}=y\log y+ye^x$. If $y(0)=1$, then $y(1)=$

(a) $e^e$ (b) $e^{-e}$ (c) $e^{1/e}$ (d) $\sqrt{e}$

**8.** The function $f(\lambda)=\int_0^{\pi/2}\frac{dx}{\sqrt{1-\lambda\cos^2 x}}$, $\lambda\in(0,1)$ is

(a) increasing (b) decreasing

(c) increasing on (0, 1/2) and decreasing on (1/2, 1)

(d) increasing on (1/2, 1) and decreasing on (0, 1/2)

**9.** Let $f$ satisfy the equation $x=f(x)\cdot e^{f(x)}$ then $\int_0^e f(x)\,dx=$

(a) 1 (b) $e-1$ (c) $e+1$ (d) 0

**10.** Let $a, b\in R, a<b$ and let $f$ be a differentiable function on the interval $[a, b]$ then $\lim_{n\to\infty}\int_a^b f(x)\sin nx\,dx=$

(a) 0 (b) 1 (c) $b$ (d) $b-a$

**11.** The shortest possible length of an interval $[a, b]$ for which $\int_a^b \frac{4}{1+x^2}dx=\pi$ is

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $2\sqrt{2}+2$ (d) $2\sqrt{2}-2$

## SECTION-II

### Comprehension Type

**Paragraph for Question No. 12-14**

Let $\theta_k(x)$ be 0 for $x<k$ and 1 for $x\ge k$. The dirac delta function is defined to be $\delta_k(x)=\frac{d}{dx}\theta_k(x)$. Suppose

By : Tapas Kr. Yogi, Mob : 9533632105.

$$\frac{d^2 f(x)}{dx^2} = \delta_1(x) + \delta_2(x) \text{ and } f(0) = f'(0) = 0.$$

**12.** The value of $f(5)$ is
(a) 2 (b) 5 (c) 7 (d) 10

**13.** The value of $f'(5)$ is
(a) 2 (b) 5 (c) 7 (d) 10

**14.** The number of points where $f'(x)$ is not differentiable is
(a) zero (b) one (c) two (d) infinite

**Paragraph for Question No. 15-17**

For the curve $y = \frac{(x-3y)^{3/2}}{\sqrt{x}}$, $\frac{dy}{dx} = \frac{y}{Q(x)}$

**15.** $y^2 + (Q(x))^2 = \log\pi$ represents
(a) line (b) circle
(c) sine curve (d) ellipse

**16.** $Q'(1) =$
(a) 0 (b) –1 (c) 1 (d) –2

**17.** $\lim_{x \to 0} \frac{\sin(Q(x))}{x} =$
(a) 0 (b) 1 (c) –1 (d) does not exist

## SOLUTIONS

**1. (b) :** Rewrite the given integral eqn. as
$$\int_0^1 2xf(x^2)\,dx = \int_0^1 x^2 dx + \int_0^1 f^2(x^2)\,dx$$
*i.e.* $\int_0^1 (f(x^2) - x)^2 dx = 0$ *i.e.,* $f(x^2) = x$

**2. (d) :** Rewrite the given integral as
$$\int_0^1 xf(x) - f^2(x)\,dx = \int_0^1 \frac{x^2}{4}\,dx$$
*i.e.,* $\int_0^1 \left(f(x) - \frac{x}{2}\right)^2 dx = 0 \Rightarrow f(x) = \frac{x}{2}$

**3. (a) :** Use property, $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

**4. (d) :** Use the standard result, $\frac{d^2x}{dy^2} = \frac{-y_2}{y_1^3}$, the given equation reduces to $y_2 - \frac{y_2}{y_1^3} = 0$ *i.e.* $y_2 = 0$ or $y_1 = 1$

$y_2 = 0 \Rightarrow y_1 = b$ and $y = bx + c$ (linear function)
$y_1 = 1 \Rightarrow y = x + k$ (which is already included in the above solution)
Hence, $y = bx + c$, $b \neq 0$

**5. (b) :** Let $I = \int_0^1 f(x)\,dx = \int_0^{\pi/2} f(\sin\theta)\cos\theta d\theta$

or, $I = \int_0^{\pi/2} f(\cos\theta)\sin\theta d\theta$ [By $(a-x)$ property]

Adding, we get

$2I \le \int_0^{\pi/2} 1\,d\theta$ *i.e.* $I \le \frac{\pi}{4}$

**6. (a) :** Notice that $\int_0^\theta \sqrt{1+\cos^2 t}\,dt$ is the arc length of the curve $y = \sin x$ from (0, 0) to ($\theta$, $\sin\theta$) and $\sqrt{\theta^2 + \sin^2\theta}$ is the distance between these points.

**7. (a) :** Put $y = e^t$ to convert the given differential equation into linear equation.

**8. (a) :** For $x \in (0, \pi/2)$, $0 \le \lambda_1 < \lambda_2 < 1$
$-\lambda_1\cos^2 x > -\lambda_2\cos^2 x$
$\Rightarrow f(\lambda_1) < f(\lambda_2) \Rightarrow$ increasing function

**9. (b) :** Note that $f$ is inverse function of $g(y) = ye^y$ and $f(e) = 1$

So, $\int_0^e f(x)dx + \int_0^1 g(y)dy = e(1) - 0(0)$

$\Rightarrow \int_0^e f(x)dx = e - \int_0^1 ye^y dy = e - 1$

**10. (a) :** $\int_a^b f(x)\sin nx dx = \frac{1}{n}[f(a)\cos na - f(b)\cos nb]$
$$+ \frac{1}{n}\cdot\int_a^b f'(x)\cos nx dx$$
(Integration by parts)

Now, $\left|\frac{1}{n}(f(a)\cos na - f(b)\cos nb\right| \le \frac{1}{n}(|f(a)\cos na|$
$+ |f(b)\cos nb|)$
(triangle inequality)

$$\le \frac{1}{n}(|f(a)| + |f(b)|) \to 0$$

## MPP-4 CLASS XII ANSWER KEY

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | (d) | **2.** | (b) | **3.** | (d) | **4.** | (d) | **5.** | (c) |
| **6.** | (b) | **7.** | (a,b,c) | **8.** | (a,c) | **9.** | (a,b,c) | **10.** | (a,d) |
| **11.** | (b,c) | **12.** | (a,d) | **13.** | (a,c) | **14.** | (b) | **15.** | (b) |
| **16.** | (b) | **17.** | (6) | **18.** | (6) | **19.** | (3) | **20.** | (3) |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and $\left|\frac{1}{n}\int_a^b f'(x)\cos nx\,dx\right| \le \frac{1}{n}\cdot\int_a^b |f'(x)||\cos nx|\,dx$

$$\le \frac{1}{n}\int_a^b M\,dx = \frac{M(b-a)}{n} \to 0$$

where $|f'(x)| \le$ some $M$

Hence, $\int_a^b f(x)\sin nx\,dx \to 0$ as $n \to \infty$

**11. (d) :** $\tan^{-1}b - \tan^{-1}a = \frac{\pi}{4} \Rightarrow b = \frac{a+1}{1-a}$

So, length of interval $l = b - a = \frac{1+a^2}{1-a}$

Using, $\frac{dl}{da} = 0$, at $l_{min}$, we have $a = 1-\sqrt{2}$

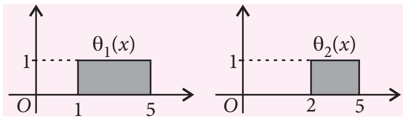
**12. (c):** When integrated once,

$f'(x) = \int \delta_1(x) + \delta_2(x)\cdot dx = \theta_1(x) + \theta_2(x) + c$

$f'(0) = c = 0$

So, $f'(x) = \theta_1(x) + \theta_2(x)$

To find $f(x)$, weintegrate $f(x)$ or take the area covered by $\theta_1(x)$ and $\theta_2(x)$.



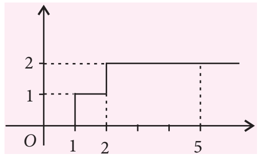

Now the graphs of $\theta_1(x)$ and $(\theta_2)x$ are

So, $f(5)$ = area of the shaded region

$= (4 \times 1) + (3 \times 1) = 7$

**13. (a) :** Combine the graphs of $\theta_1(x)$ and $\theta_2(x)$ to get the graph of $f'(x)$.

So, $f'(5) = 2$




**14. (c):** From the graph, it is clear that $f'(x)$ is not differetiable at two points $x = 1$ and $x = 2$.

**15. (b) :** The given curve equation is $xy^2 = (x - 3y)^3$, a homogeneous equation.

So, $\frac{dy}{dx} = \frac{y}{x}$ *i.e.* $Q(x) = x$

So, $y^2 + (Q(x))^2 = \log\pi$ represents a circle.

**16. (c):** $Q'(x) = 1 \Rightarrow Q'(1) = 1$

**17. (b) :** $\lim_{x\to 0} \frac{\sin Q(x)}{x} = \lim_{x\to 0}\frac{\sin x}{x} = 1$

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# MPP-4 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

## Application of Derivatives

**Total Marks : 80** **Time Taken : 60 Min.**

### Only One Option Correct Type

**1.** The equation of the tangent to the curve $y = e^{-|x|}$ at the point where the curve cuts the line $x = 1$ is

(a) $x + y = e$ (b) $e(x + y) = 1$

(c) $y + ex = 1$ (d) none of these

**2.** The acute angles between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection is

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

(c) $\tan^{-1}(4\sqrt{7})$ (d) none of these

**3.** If $f(x) = x^{\alpha} \ln x$ and $f(0) = 0$, then the possible value of $\alpha$ for which Rolle's theorem can be applied in $[0, 1]$ is

(a) $-2$ (b) $-1$

(c) $0$ (d) none of these

**4.** If $f: R \to R$ is the function defined by

$$f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}, \text{ then}$$

(a) $f(x)$ is an increasing function

(b) $f(x)$ is a decreasing function

(c) $f(x)$ is onto (surjective)

(d) none of these

**5.** Let $f(x) = \tan x - \tan 2x$. Then

(a) $f$ has no critical point

(b) $f$ has minimum at $x = \cos^{-1}(\sqrt{10} - 1)/4$

(c) $f$ has minimum at $x = \cos^{-1}(\sqrt{10} - \sqrt{2})/4$

(d) $f$ has maximum at $x = \cos^{-1}(\sqrt{10} - \sqrt{2})/4$

**6.** The diagonal of a square is changing at the rate of 0.5 cm/sec. Then the rate of change of area, when the area is 400 $\text{cm}^2$, is equal to

(a) $20\sqrt{2}\ \text{cm}^2/\text{sec}$ (b) $10\sqrt{2}\ \text{cm}^2/\text{sec}$

(c) $\frac{1}{10\sqrt{2}}\ \text{cm}^2/\text{sec}$ (d) $\frac{10}{\sqrt{2}}\ \text{cm}^2/\text{sec}$

### One or More Than One Option(s) Correct Type

**7.** A point on the ellipse $4x^2 + 9y^2 = 36$, where the tangent is equally inclined to the axes is

(a) $\left(\frac{9}{\sqrt{13}}, \frac{4}{\sqrt{13}}\right)$ (b) $\left(-\frac{9}{\sqrt{13}}, \frac{4}{\sqrt{13}}\right)$

(c) $\left(\frac{9}{\sqrt{13}}, -\frac{4}{\sqrt{13}}\right)$ (d) none of these

**8.** The extreme values of the function

$$f(x) = \frac{1}{\sin x + 4} - \frac{1}{\cos x - 4} \text{ where } x \in R \text{ is}$$

(a) $\frac{4}{8 - \sqrt{2}}$ (b) $\frac{2\sqrt{2}}{8 - \sqrt{2}}$

(c) $\frac{2\sqrt{2}}{4\sqrt{2} + 1}$ (d) $\frac{4\sqrt{2}}{8 + \sqrt{2}}$

**9.** If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$, then

(a) $f(x)$ is increasing on $[-1, 2]$

(b) $f(x)$ is continuous on $[-1, 3]$

(c) $f'(2)$ does not exist

(d) $f(x)$ has the maximum value at $x = 2$

10. Let $f(x)=\begin{cases}-x^3+\dfrac{b^3-b^2+b-1}{b^2+3b+2}, & 0\le x<1\\ 2x-3, & 1\le x<3\end{cases}$

If $f(x)$ has least value at $x = 1$, then

(a) $-2 < b < -1$ (b) $-1 < b < 0$

(c) $0 < b < 1$ (d) $1 \le b < \infty$

11. Let the function $f(x) = \sin x + \cos x$, be defined in $[0, 2\pi]$, then $f(x)$

(a) increases in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(b) decreases in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(c) increases in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

(d) decreases in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, 2\pi\right]$

12. Let $f(x) = 2x^2 - \ln|x|$, $x \ne 0$, then $f(x)$ is monotonically

(a) increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(b) decreasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(c) increasing in $\left(-\infty, \frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

(d) decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

13. Rolle's theorem holds for the function

$f(x) = x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point $\frac{4}{3}$, then

(a) $c = 8$ (b) $c = -5$

(c) $b = -5$ (d) $b = 8$

## Comprehension Type

If $f(x) = |x - 1| + |x - 3| + |5 - x|$, $\forall\ x \in R$

14. If $f(x)$ increases, then $x \in$

(a) $(1, \infty)$ (b) $(3, \infty)$ (c) $(5, \infty)$ (d) $(1, 3)$

15. If $f(x)$ decreases, then $x \in$

(a) $(-\infty, 1)$ (b) $(-\infty, 3)$

(c) $(-\infty, 5)$ (d) $(3, 5)$

## Matrix Match Type

16. Match the columns:

| | Column I | | Column II |
|---|---|---|---|
| (P) | $f(x) = \cos\pi x + 10x + 3x^2 + x^3$, $-2 \le x \le 3$. The absolute minimum value of $f(x)$ is | (1) | 3/4 |
| (Q) | If $x \in [-1, 1]$, then the minimum value of $f(x) = x^2 + x + 1$, is | (2) | 2 |
| (R) | Let $f(x) = (4/3)x^3 - 4x$, $0 \le x \le 2$. Then, the global minimum value of the function is | (3) | $-15$ |
| (S) | Let $f(x) = 6 - 12x + 9x^2 - 2x^3$, $1 \le x \le 4$. Then the absolute maximum value of $f(x)$ in the interval is | (4) | $-8/3$ |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 2 | 1 | 3 | 4 |
| (b) | 3 | 1 | 4 | 2 |
| (c) | 3 | 2 | 1 | 4 |
| (d) | 1 | 2 | 3 | 4 |

## Integer Answer Type

17. If the approximate value of $\log_{10}(4.04)$ is $0.abcdef$. It is given that $\log_{10} 4 = 0.6021$ and $\log_{10} e = 0.4343$, then the value of $a$ must be

18. The minimum value of the expression

$\frac{3b+4c}{a} + \frac{4c+a}{3b} + \frac{a+3b}{4c}$ ($a, b, c$ are +ve) is

19. The number of critical points of the function $f'(x)$ where $f(x) = \frac{|x-2|}{x^2}$ is

20. The three sides of a trapezium are equal each being 6 cm long. If area of trapezium when it is maximum is $27\sqrt{A}$, then the value of $A$ must be ◆◆

*Keys are published in this issue. Search now!* ☺

**SELF CHECK**

No. of questions attempted ......

No. of questions correct ......

Marks scored in percentage ......

**Check your score! If your score is**

| | | |
|---|---|---|
| > 90% | EXCELLENT WORK ! | You are well prepared to take the challenge of final exam. |
| 90-75% | GOOD WORK ! | You can score good in the final exam. |
| 74-60% | SATISFACTORY ! | You need to score more next time. |
| < 60% | NOT SATISFACTORY! | Revise thoroughly and strengthen your concepts. |

# MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

## PROBLEM Set 166

### JEE MAIN

1. If $f(x, y)$ is a polynomial of degree 3 such that $f(0, 0) = f(\pm 1, 0) = f(0, \pm 1) = f(2, 2) = 0$, then $f(a, b) = 0$ for $a + b =$

(a) $\frac{21}{20}$ (b) $\frac{19}{21}$ (c) $\frac{21}{19}$ (d) $\frac{20}{21}$

2. The inradius of a right angled triangle with integer sides is 2011. The number of such triangles is

(a) 2 (b) 3 (c) 5 (d) 7

3. The radius of the largest circle with centre $\left(\frac{1}{2}, 0\right)$ inscribed in the ellipse $x^2 + 2y^2 = 2$ is

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

4. In triangle $ABC$, $C = \frac{\pi}{2}$, $D$ and $E$ are points on the side $AC$ such that $AD = 11$, $DE = 5$.

If $\angle CAB : \angle CDB : \angle CEB = 1 : 2 : 3$, then $BC =$

(a) 8 (b) 9 (c) $\frac{44}{5}$ (d) $\frac{15}{2}$

5. The normal to the curve $2y + 5x^5 - 10x^3 + x + 6 = 0$ at the point $(0, -3)$ is a tangent to the curve at the point

(a) $(2, -44)$ (b) $(-2, 38)$
(c) $(1, -1)$ (d) $(-1, -4)$

### JEE ADVANCED

6. If $\frac{N}{1!\,14!} = \sum_{r=2}^{7} \frac{1}{r!(15-r)!}$, then $5N$ is divisible by

(a) 2 (b) 3 (c) 11 (d) 31

### COMPREHENSION

Consider 5-digit numbers formed using the digits 0, 1, 2, 3, 4, 5 without repetition of digits.

7. The number of numbers divisible by 4 is

(a) 48 (b) 54 (c) 66 (d) 144

8. The number of numbers divisible by 12 is

(a) 54 (b) 66 (c) 108 (d) 144

### INTEGER MATCH

9. If $y = f(x)$ is the orthogonal trajectory of the circles $(x - c)^2 + y^2 = 1, x \geq 0, f(0) = 1$ and $f(x) = \frac{\sqrt{3}}{2}$, then $e^{2x+1}$ is

### MATRIX MATCH

10. Match the following columns.

| | Column I | | Column II |
|---|---|---|---|
| (P) | $\int_{1/e}^{\tan x} \frac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)} =$ | (1) | $\frac{1}{4}$ |
| (Q) | $\int_0^{\pi/2} \frac{dx}{(\sqrt{\sin x} + \sqrt{\cos x})^4} =$ | (2) | $\frac{1}{3}$ |
| (R) | $I_1 = \int_0^1 \frac{\tan^{-1} x}{x} dx$ and $I_2 = \int_0^{\pi/2} \frac{x}{\sin x} dx$, $\frac{I_1}{I_2} =$ | (3) | $\frac{1}{2}$ |
| (S) | If $f(x) = \int_1^x \frac{\ln t}{1+t} dt$ and $F(x) = f(x) + f\left(\frac{1}{x}\right)$, then $F(e) =$ | (4) | 1 |
| | | (5) | 2 |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 4 | 2 | 1 | 5 |
| (b) | 4 | 2 | 3 | 3 |
| (c) | 3 | 4 | 2 | 3 |
| (d) | 5 | 1 | 2 | 4 |

See Solution Set of Maths Musing 165 on page no. 84

# beat the
# TIME TRAP


**Duration : 30 minutes**

## SECTION-I
### Single Correct Answer Type

**1.** If $A=\sum_{r=0}^{\infty}\left\{\left(-\frac{1}{2}\right)^r \sin^{2r}x\right\}, B=\sum_{r=0}^{\infty}\sin^{2r}x,$ then number of solution in $[-2\pi, 2\pi]$ of $A:B = 4\sin^2 x : (1+\cos 2x)$ is

(a) 2 (b) 4 (c) 8 (d) none of these

**2.** $\lim_{x\to\infty}\left[\frac{\sum_{r=1}^{2013}(r+x)^{2013}}{x^{2013}+2013^{2013}}\right]=$

(a) 2014 (b) 2013 (c) $\frac{1}{2013}$ (d) none of these

**3.** If $\int\frac{dx}{\sqrt[3]{x^2}-\sqrt[3]{x}}=\alpha\left[\sqrt[\gamma]{x}+\log_e|\sqrt[\gamma]{x}-1|\right]+c$, then

(a) $\alpha, \beta, \gamma$ are in A.P.
(b) $\alpha, \beta, \gamma$ are in G.P.
(c) $\alpha=\beta=\gamma$ (d) all of these

**4.** If $f(x)$ be an identity function, then equation $\sum_{r=1}^{3}\{f(x)-(2012+r)\}^{-1}=0$ has

(a) no real roots (b) real and equal roots
(c) real and different roots
(d) none of these

**5.** If $\sin x : \sin y : \sin z = \cos A : \cos B : \cos C$ then $\sum\left(\frac{\sin^2 A-\cos^2 x}{\sin x-\cos A}\right)=$

(a) $\frac{\sum\sin^2 x-\sum\cos^2 A}{\left(\sum\sin x\right)\left(\sum\cos A\right)}$ (b) $\frac{\sum\sin A\cos x}{\sum\sin x-\sum\cos A}$

(c) $\frac{\sum(\sin A-\cos x)^2}{\sum\sin A\cos x}$

(d) $\frac{\left(\sum\sin x\right)^2-\left(\sum\cos A\right)^2}{\sum(\sin x-\cos A)}$

**6.** If $\sum_{r=0}^{\infty}\left(x^r+x^{r+1}+x^{r+2}\right)=\sum_{r=0}^{\infty}p_r x^r\ (|x|<1),$ then $\sum_{r=0}^{671}p_r=$

(a) 0 (b) 2012 (c) 2015 (d) 2013

**7.** If $\int(1+x\tan x)^{-2}dx=\frac{1}{x+f(x)}+c,$ then $f(x)=$

(a) $x\tan x$ (b) $\cot x$
(c) $\tan x$ (d) none of these

**8.** If $x\sin^2\alpha + y + z = 0$, $x + y\sin^2\beta + z = 0$ & $x + y + z\sin^2\gamma = 0$ ($\alpha \ne \beta \ne \gamma \ne (2n+1)\pi/2$, where $n \in I$) have a non-trivial solution, then $\sum\tan^2\alpha =$

(a) 0 (b) 1 (c) 2 (d) none of these

**9.** If $a_i > 0$ ($i = 1, 2, 3, 4$) so that $502a_1 + 503a_2 + 504a_3 + 505a_4 = 2014$ and $256\, a_1a_2a_3a_4 \ge \left(\sum_{r=1}^{4}a_r\right)^4$, then $\sum_{r=1}^{4}a_r^r=$

(a) 2014 (b) 1 (c) 4 (d) none of these

**10.** If $x$ and $y$ be two real variables satisfying $x^2+y^2=\frac{t^2-1}{t}$ and $x^4+y^4=\frac{t^4+1}{t^2}$, then which of the following is(are) true?

(a) $y^2 + x^{-2} = 0$ (b) $x^3 y=\frac{dx}{dy}$
(c) $xdy + ydx = 0$ (d) none of these

## SECTION-II

### Multiple Correct Answer Type

**11.** If $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a} \perp \vec{b}, c$ is inclined at the same angle to both $\vec{a}$ and $\vec{b}$ and $\vec{c} = p\vec{a} + q\vec{b} + r(\vec{a} \times \vec{b})$ then

(a) $p = q$ (b) $|p| \le 1$

(c) $|q| \le 1$ (d) $pq > 1$

**12.** If $a$, $b$, $c$ be three positive numbers and $(ab + bc + ca)x^2 + (a + b + c)x + 1 = 0$ has complex roots, then

(a) $\sqrt{a} + \sqrt{c} > \sqrt{b}$

(b) $\sqrt{a} + \sqrt{b} + \sqrt{c} > 4$

(c) $\left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} - \frac{1}{\sqrt{ca}}\right)\left(\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} - \frac{1}{\sqrt{ab}}\right) \times \left(\frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} - \frac{1}{\sqrt{bc}}\right) > 0$

(d) none of these

**13.** If $a \sin\theta + b \cos\theta = c = a \operatorname{cosec}\theta + b \sec\theta$, then

(a) $\sin 2\theta = \frac{2ab}{c^2 - a^2 - b^2}$

(b) $\tan^3\theta = \frac{a}{b}$

(c) $a \cos^3\theta + b \sin^3\theta = 0$

(d) $\sin\theta + \operatorname{cosec}\theta = \frac{a^2 + c^2 - b^2}{ac}$

**14.** Let $f(x) = x^{\frac{1}{\{x\}}}$ (where $\{\cdot\}$ denotes the fractional part of $x$) then

(a) $\lim_{x \to 0^+} f(2013^x) = 2013$

(b) $\lim_{x \to 0^-} f(2013^x) = 1$

(c) $\lim_{x \to 0^+} f(2013^x) = e$

(d) $\lim_{x \to 0} f(2013^x)$ does not exist

**15.** A person draws 3 balls randomly from a bag containing 3 white and 3 black balls and then he put 3 red balls into the bag and draws 3 balls again randomly. The probability that now he has all 3 balls of different colour is

(a) > 20 % (b) > 25 %

(c) > 30 % (d) > 33 %

## SOLUTIONS

**1. (c) :**

$$A = \sum_{r=0}^{\infty}\left\{\left(-\frac{1}{2}\right)^r \sin^{2r} x\right\} = 1 - \frac{1}{2}\sin^2 x + \frac{1}{4}\sin^4 x - \ldots \text{ to} \infty$$

$$= \frac{1}{1 - \left(-\frac{1}{2}\sin^2 x\right)} = \frac{2}{2 + \sin^2 x}$$

$$B = \sum_{r=0}^{\infty} \sin^{2r} x = 1 + \sin^2 x + \sin^4 x + \ldots \text{to} \infty = \frac{1}{\cos^2 x}$$

$$\therefore\ A : B = 4\sin^2 x : (1 + \cos 2x) \Rightarrow \frac{2\cos^2 x}{2 + \sin^2 x} = \frac{4\sin^2 x}{2\cos^2 x}$$

$$\Rightarrow\ \cos^4 x = 2\sin^2 x + \sin^4 x \Rightarrow \sin x = \pm\frac{1}{2}$$

Hence, there will be 8 solutions in $[-2\pi, 2\pi]$

**2. (b):**

$$\lim_{x \to \infty} \frac{\{(1+x)^{2013} + (2+x)^{2013} + \ldots + (2013+x)^{2013}\}}{x^{2013} + 2013^{2013}}$$

$$= \lim_{x \to \infty} \frac{x^{2013}\left\{\left(\frac{1}{x}+1\right)^{2013} + \left(\frac{2}{x}+1\right)^{2013} + \ldots\ldots\ldots + \left(\frac{2013}{x}+1\right)^{2013}\right\}}{x^{2013}\left(1 + \left(\frac{2013}{x}\right)^{2013}\right)}$$

$$= \frac{(0+1)^{2013} + (0+1)^{2013} + \ldots + (0+1)^{2013}}{1+0} = 2013$$

[As $x \to \infty$, $1/x \to 0$]

**3. (d):** Let $x = z^3 \Rightarrow dx = 3z^2 dz$

$$\int \frac{3z^2 dz}{z^2 - z} = 3\int \frac{z dz}{z-1} = 3\int \frac{(z-1)+1}{z-1} dz$$

$$= 3\left[z + \log|z-1|\right] + c = 3\left[\sqrt[3]{x} + \log|\sqrt[3]{x} - 1|\right] + c$$

$\Rightarrow\ \alpha = \beta = \gamma = 3 \Rightarrow \alpha, \beta, \gamma$ are in A.P. and G.P. both.

**4. (c) :** $\because f(x)$ is an identity function $\therefore f(x) = x$

$\therefore$ Given equation becomes

$$\sum_{r=1}^{3} \{x - (2012 + r)\}^{-1} = 0$$

$$\Rightarrow\ \frac{1}{x-2013} + \frac{1}{x-2014} + \frac{1}{x-2015} = 0$$

$\Rightarrow$ $(x-2014)(x-2015)+(x-2015)(x-2013)$
$+(x-2013)(x-2014)=0 \Rightarrow g(x)=0$ [say]

$\therefore$ $g(2013)=(-1)(-2)=2>0$, $g(2014)=(-1)(1)=-1<0$ and $g(2015)=(2)(1)=2>0$

As $g(x)$ changes sign between 2013 & 2014, 2014 & 2015, the equation $g(x)=0$ has roots between them.

$\therefore$ Both roots are real and different.

**5. (d):** Let $\dfrac{\sin x}{\cos A}=\dfrac{\sin y}{\cos B}=\dfrac{\sin z}{\cos C}=k$

$\therefore$ $\sin x = k\cos A$ ...(i)

$$\therefore \sum\left(\frac{\sin^2 A-\cos^2 x}{\sin x-\cos A}\right)=\sum\left(\frac{1-\cos^2 A-1+\sin^2 x}{\sin x-\cos A}\right)$$

$$=\sum\left(\frac{\sin^2 x-\cos^2 A}{\sin x-\cos A}\right)$$

$=\Sigma(\sin x+\cos A)=\Sigma\sin x+\Sigma\cos A$

$$=\frac{\left(\sum\sin x+\sum\cos A\right)\left(\sum\sin x-\sum\cos A\right)}{\sum\sin x-\sum\cos A}$$

$$=\frac{\left(\sum\sin x\right)^2-\left(\sum\cos A\right)^2}{\sum(\sin x-\cos A)}$$

**6. (d):** $\sum_{r=0}^{\infty} x^r(1+x+x^2)=\sum_{r=0}^{\infty} p_r x^r$

$\Rightarrow$ $(1+x+x^2)(1+x+x^2+x^3+\ldots \text{ to } \infty)$
$=p_0+p_1x+p_2x^2+p_3x^3+\ldots \text{ to } \infty$

$\Rightarrow$ $(1+x+x^2)\left(\dfrac{1}{1-x}\right)=p_0+p_1x+p_2x^2+\ldots+\text{to }\infty$

$\Rightarrow$ $p_0+p_1x+p_2x^2+p_3x^3+\ldots \text{ to } \infty$
$=(1-x)^{-2}(1-x^3)$
$=(1+2x+3x^2+4x^3+\ldots \text{ to } \infty)(1-x^3)$

$\therefore$ On comparing, we get
$p_0=1, p_1=2, p_2=3=p_3=p_4=p_5=\ldots\ldots$

$\therefore$ $\sum_{r=0}^{671} p_r=p_0+p_1+p_2+p_3+\ldots+p_{671}$
$=1+2+(3+3+3+\ldots \text{ to } 670 \text{ terms})$
$=3\times 671=2013$

**7. (b):** Let $I=\displaystyle\int\frac{dx}{(1+x\tan x)^2}$

$$=\int\frac{\cot^2 x\,dx}{(\cot x+x)^2}=\int\frac{-(1-\operatorname{cosec}^2 x)dx}{(\cot x+x)^2}$$

$$=-\int\frac{d(x+x\cot x)}{(x+\cot x)^2}=\frac{1}{x+\cot x}+c \Rightarrow f(x)=\cot x$$

**8. (d):** For non-trivial solution, we must have

$$\begin{vmatrix}\sin^2\alpha & 1 & 1\\ 1 & \sin^2\beta & 1\\ 1 & 1 & \sin^2\gamma\end{vmatrix}=0 \Rightarrow \begin{vmatrix}p & 1 & 1\\ 1 & q & 1\\ 1 & 1 & r\end{vmatrix}=0$$

(where $p=\sin^2\alpha, q=\sin^2\beta, r=\sin^2\gamma$)

$$\Rightarrow \begin{vmatrix}p & 1 & 1\\ 1-p & q-1 & 0\\ 1-p & 0 & r-1\end{vmatrix}=0\begin{bmatrix}R_2\to R_2-R_1,\\ R_3\to R_3-R_1\end{bmatrix}$$

$\Rightarrow$ $p(q-1)(r-1)-(1-p)(r-1)+1\{0-(1-p)(q-1)\}=0$

$$\Rightarrow \frac{(p-1)+1}{p-1}+\frac{1}{q-1}+\frac{1}{r-1}=0$$

$$\Rightarrow \frac{1}{1-\sin^2\alpha}+\frac{1}{1-\sin^2\beta}+\frac{1}{1-\sin^2\gamma}=1$$

$$\Rightarrow \sec^2\alpha+\sec^2\beta+\sec^2\gamma=1\Rightarrow\sum(1+\tan^2\alpha)=1$$

$\Rightarrow$ $\sum\tan^2\alpha=-2<0$, which is impossible

**9. (c) :** $\because$ $256\, a_1a_2a_3a_4\ge\left(\sum_{r=1}^{4}a_r\right)^4$

$\because$ $4\sqrt[4]{a_1a_2a_3a_4}\ge a_1+a_2+a_3+a_4$

$\Rightarrow$ $\dfrac{a_1+a_2+a_3+a_4}{4}\le\sqrt[4]{a_1a_2a_3a_4}\Rightarrow$ A.M.$\le$ G.M.

But, A.M $\ge$ G.M $\Rightarrow$ A.M = G.M

$\Rightarrow$ $a_1=a_2=a_3=a_4$ ...(i)

Also, $502a_1+503a_2+504a_3+505\,a_4=2014$

$\therefore$ $2014\,a_1=2014\Rightarrow a_1=1=a_2=a_3=a_4$

Now, $\sum_{r=1}^{4}a_r^r=a_1+a_2^2+a_3^3+a_4^4=1+1+1+1=4$

**10. (d):** $\because$ $x^2+y^2=t-\dfrac{1}{t}$ and $x^4+y^4=t^2+\dfrac{1}{t^2}$

$\Rightarrow$ $x^4+y^4+2x^2y^2=t^2+\dfrac{1}{t^2}-2$

and $x^4+y^4=t^2+\dfrac{1}{t^2}$

$\Rightarrow$ $x^2y^2=-1$, which is not possible for $x, y\in R$.

$\therefore$ (a), (b) and (c) are not possible.

**11. (a,b,c):** $\because$ $|\vec a|=|\vec b|=|\vec c|=1$

Let $\theta$ be the angle between $\vec a\,\&\,\vec c$ and $\vec b\,\&\,\vec c$

$\therefore$ $\cos\theta=\dfrac{\vec a\cdot\vec c}{|\vec a||\vec c|}=\vec a\cdot\vec c$ ...(i)

Similarly, $\cos\theta = \vec{b}\cdot\vec{c}$ ...(ii)

$\because \quad \vec{a}\perp\vec{b} \quad \therefore \quad \vec{a}\cdot\vec{b}=0$

Also, $\vec{c} = p\vec{a}+q\vec{b}+r(\vec{a}\times\vec{b})$

$\therefore \quad \vec{a}\cdot\vec{c} = p\vec{a}\cdot\vec{a}+q\,\vec{a}\cdot\vec{b}+r\,\vec{a}\cdot(\vec{a}\times\vec{b})$

$= p + 0 + 0 \; = p \Rightarrow \cos\theta = p \;\; \therefore \;\; |p| = |\cos\theta| \le 1$

Similarly, $\cos\theta = q \Rightarrow |q| \le 1$

Also, $p = q$.

**12. (a, c) :** $\because$ Roots are complex

$\therefore \quad D < 0 \Rightarrow (a + b + c)^2 - 4\,(ab + bc + ca) < 0$

$\Rightarrow \quad (a + c)^2 + b^2 + 2b(a + c) - 4b(a + c) < 4\,ac$

$\Rightarrow \quad (a + c - b)^2 < 4\,ac \;\Rightarrow\; -2\sqrt{ac} < a+c-b < 2\sqrt{ac}$

$\Rightarrow \quad (\sqrt{a}+\sqrt{c})^2 > b \Rightarrow \sqrt{a}+\sqrt{c} > \sqrt{b} \Rightarrow$ (a) is correct.

$$\Rightarrow \quad \sqrt{a}+\sqrt{c}-\sqrt{b}>0 \;\Rightarrow\; \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ab}}-\frac{1}{\sqrt{ca}}>0$$

Similarly,

$$\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}-\frac{1}{\sqrt{ab}}>0 \text{ and } \frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}-\frac{1}{\sqrt{bc}}>0$$

$\therefore \quad$ On multiplying,

$$\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}-\frac{1}{\sqrt{ca}}\right)\left(\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}-\frac{1}{\sqrt{ab}}\right)\left(\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}-\frac{1}{\sqrt{bc}}\right)>0$$

$\Rightarrow \quad$ (c) is correct.

**13. (a, c, d) :** $\because a\sin\theta + b\cos\theta = c$ ...(i)

& $a\,\mathrm{cosec}\,\theta + b\sec\theta = c$ ...(ii)

$$\therefore \quad \text{On multiplying, } a^2+b^2+ab\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)=c^2$$

$$\Rightarrow \quad a^2+b^2+2ab\left(\frac{1}{\sin 2\theta}\right)=c^2 \;\therefore\; \sin 2\theta=\frac{2ab}{c^2-a^2-b^2}$$

From (i) & (ii), $a\sin\theta + b\cos\theta = a\,\mathrm{cosec}\,\theta + b\sec\theta$

$$\Rightarrow a\left(\frac{1}{\sin\theta}-\sin\theta\right)+b\left(\frac{1}{\cos\theta}-\cos\theta\right)=0$$

$$\Rightarrow a\frac{\cos^2\theta}{\sin\theta}+b\frac{\sin^2\theta}{\cos\theta}=0 \;\therefore\; a\cos^3\theta+b\sin^3\theta=0$$

From (i), $b\cos\theta = c - a\sin\theta$ ...(iii)

& from (ii), $b\sec\theta = c - a\,\mathrm{cosec}\,\theta$ ...(iv)

From (iii) & (iv), $b^2 = c^2 + a^2 - ac\,(\sin\theta + \mathrm{cosec}\theta)$

$$\therefore \;\; \sin\theta+\mathrm{cosec}\,\theta=\frac{a^2+c^2-b^2}{ac}$$

**14. (b,c,d) :**

$\because$ If $x \to 0^+$, $2013^x \in (1, 2) \Rightarrow \{2013^x\} = 2013^x - 1$

And, $x \to 0^-$, $2013^x \in (0, 1) \Rightarrow \{2013^x\} = 2013^x$

$$\lim_{x\to 0^+} f(2013^x) = \lim_{x\to 0^+}(2013^x)^{\frac{1}{\{2013^x\}}}$$

$$= \lim_{x\to 0^+}(2013^x)^{\frac{1}{2013^x-1}}$$

$$= \lim_{h\to 0}(2013^h)^{\frac{1}{2013^h-1}} \quad (1^\infty \text{ form})$$

$$= e^{\lim\limits_{h\to 0}\left[\left(2013^h-1\right).\frac{1}{(2013^h-1)}\right]} = e$$

And,

$$\lim_{x\to 0^-} f(2013^x) = \lim_{x\to 0^-}\left(2013^x\right)^{\frac{1}{\{2013^x\}}}$$

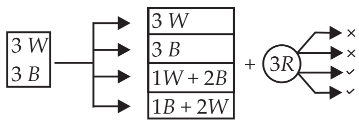
$$= \lim_{x\to 0^-}\left(2013^x\right)^{\frac{1}{2013^x}} = \lim_{h\to 0}\left(2013^{0-h}\right)^{\frac{1}{\{2013^{0-h}\}}}$$

$$= \lim_{h\to 0}\left(2013^{-h}\right)^{\frac{1}{2013^{-h}}} = 1^1 = 1$$

$\because \quad$ R.H.L $\neq$ L.H.L

$\Rightarrow \quad \lim\limits_{x\to 0} f(2013^x)$ does not exist

**15. (a, b) :** Diagramatically




Clearly, required probability

$$= 2\left[\left(\frac{{}^3C_1\times{}^3C_2}{{}^6C_3}\right)\times\left(\frac{{}^1C_1\times{}^2C_1\times{}^3C_1}{{}^6C_3}\right)\right]$$

$$= \frac{2\cdot 3\cdot 3\cdot 1\cdot 2\cdot 3}{20\cdot 20} = \frac{27}{100} = 27\%$$

# OLYMPIAD CORNER

1. Suppose a quadratic function $f(x) = ax^2 + bx + c$ ($a$, $b$, $c \in R$ and $a \neq 0$) satisfies the following conditions :
(1) When $x \in R$, $f(x - 4) = f(2 - x)$ and $f(x) \geq x$.
(2) When $x \in (0, 2)$, $f(x) \leq \left(\frac{x+1}{2}\right)^2$
(3) The minimum value of $f(x)$ on $R$ is 0.
Find the maximal $m(m > 1)$ such that there exists $t \in R$, $f(x + t) \leq x$ holds so long as $x \in [1, m]$.

2. Draw a tangent line of parabola $y = x^2$ at the point $A(1, 1)$. Suppose the line intersects the $x$-axis and $y$-axis at $D$ and $B$ respectively. Let point $C$ be on the parabola and point $E$ on $AC$ such that $\frac{AE}{EC} = \lambda_1$. Let point $F$ be on $BC$ such that $\frac{BF}{FC} = \lambda_2$ and $\lambda_1 + \lambda_2 = 1$. Assume that $CD$ intersects $EF$ at point $P$. When point $C$ moves along the parabola, find the equation of the trail of $P$.

3. Suppose that $\alpha$ and $\beta$ are different real roots of the equation $4x^2 - 4tx - 1 = 0$ ($t \in R$). $[\alpha, \beta]$ is the domain of the function $f(x) = \frac{2x - t}{x^2 + 1}$.
(1) Find $g(t) = \max f(x) - \min f(x)$.
(2) Prove that for $u_i \in \left(0, \frac{\pi}{2}\right)$ ($i = 1, 2, 3$),
if $\sin u_1 + \sin u_2 + \sin u_3 = 1$, then
$$\frac{1}{g(\tan u_1)} + \frac{1}{g(\tan u_2)} + \frac{1}{g(\tan u_3)} < \frac{3}{4}\sqrt{6}.$$

4. Suppose $A$, $B$, $C$ are three non-collinear points corresponding to complex numbers $z_0 = ai$, $z_1 = \frac{1}{2} + bi$, $z_2 = 1 + ci$ ($a$, $b$ and $c$ being real numbers), respectively. Prove that the curve $z = z_0\cos^4 t + 2z_1\cos^2 t \cdot \sin^2 t + z_2\sin^4 t$ ($t \in R$) shares a single common point with the line bisecting $AB$ and parallel to $AC$ in $\Delta ABC$ and find this point.

5. A sequence is formed by the following rules : $s_1 = a$, $s_2 = b$ and $s_{n+2} = s_{n+1} + (-1)^n S_n$ for all $n \geq 1$. If $a = 3$ and $b$ is an integer less than 1000, what is the largest value of $b$ for which 2015 is a member of the sequence? Justify your answer.

## SOLUTIONS

1. Since $f(x - 4) = f(2 - x)$ for $x \in R$, it is known that the quadratic function $f(x)$ has $x = -1$ as its axis of symmetry. By condition (3), we know that $f(x)$ opens upward, that is, $a > 0$.
Hence, $f(x) = a(x + 1)^2$ ($a > 0$).
By condition (1), we get $f(1) \geq 1$ and by (2), $f(1) \leq \left(\frac{1+1}{2}\right)^2 = 1$. It follows that $f(1) = 1$,
*i.e.*, $a(1 + 1)^2 = 1$. So $a = \frac{1}{4}$.
Thereby, $f(x) = \frac{1}{4}(x+1)^2$.
Since the graph of the parabola $f(x) = \frac{1}{4}(x+1)^2$ opens upward and a graph of $y = f(x + t)$ can be obtained by translating that of $f(x)$ by $t$ units. If we want the graph of $y = f(x + t)$ to lie under the graph of $y = x$ when $x \in [1, m]$ and $m$ to be maximal, then 1 and $m$ should be two roots of an equation with respect to $x$
$$\frac{1}{4}(x+t+1)^2 = x. \qquad \text{...(i)}$$
Substituting $x = 1$ into (i), we get $t = 0$ or $t = -4$.
When $t = 0$, substituting it into (i), we get $x_1 = x_2 = 1$ (in contradiction with $m > 1$).
When $t = -4$, substituting it into (i), we get $x_1 = 1$ and $x_2 = 9$ and so $m = 9$.

Moreover, when $t = -4$, for any $x \in [1, 9]$, we have always

$(x - 1)(x - 9) \le 0$

$\Leftrightarrow \frac{1}{4}(x-4+1)^2 \le x$, that is $f(x - 4) \le x$.

Therefore, the maximum value of $m$ is 9.

**2. 1st solution :**

The slope of the tangent line passing through $A$ is $y' = 2x|_{x=1} = 2$. So the equation of the tangent line $AB$ is $y = 2x - 1$.

Hence the coordinates of $B$ and $D$ are $B(0, -1)$, $D\left(\frac{1}{2}, 0\right)$. Thus $D$ is the midpoint of line segment $AB$.

Consider $P(x, y)$, $C(x_0, x_0^2)$, $E(x_1, y_1)$, $F(x_2, y_2)$.

Then by $\frac{AE}{EC} = \lambda_1$, we know $x_1 = \frac{1+\lambda_1 x_0}{1+\lambda_1}$, $y_1 = \frac{1+\lambda_1 x_0^2}{1+\lambda_1}$. From $\frac{BF}{FC} = \lambda_2$, we get $x_2 = \frac{\lambda_2 x_0}{1+\lambda_2}$, $y_2 = \frac{-1+\lambda_2 x_0^2}{1+\lambda_2}$.

Therefore the equation of line $EF$ is

$$\frac{y - \frac{1+\lambda_1 x_0^2}{1+\lambda_1}}{\frac{-1+\lambda_2 x_0^2}{1+\lambda_2} - \frac{1+\lambda_1 x_0^2}{1+\lambda_1}} = \frac{x - \frac{1+\lambda_1 x_0}{1+\lambda_1}}{\frac{\lambda_2 x_0}{1+\lambda_2} - \frac{1+\lambda_1 x_0}{1+\lambda_1}}.$$

Simplifying it, we get

$$[(\lambda_2 - \lambda_1)x_0 - (1 + \lambda_2)]y = [(\lambda_2 - \lambda_1)x_0^2 - 3]x + 1 + x_0 - \lambda_2 x_0^2. \quad \text{...(i)}$$

When $x_0 \ne \frac{1}{2}$, the equation of line $CD$ is

$$y = \frac{2x_0^2 x - x_0^2}{2x_0 - 1} \quad \text{...(ii)}$$

From (i) and (ii), we get $\begin{cases} x = \frac{x_0+1}{3}, \\ y = \frac{x_0}{3}. \end{cases}$

Eliminating $x_0$, we get the equation of the trail of point $P$ as $y = \frac{1}{3}(3x-1)^2$.

When $x_0 = \frac{1}{2}$, the equation of $EF$ is

$$-\frac{3}{2}y = \left(\frac{1}{4}\lambda_2 - \frac{1}{4}\lambda_1 - 3\right)x + \frac{3}{2} - \frac{1}{4}\lambda_2,$$ the equation of $CD$ is $x = \frac{1}{2}$. Combining them, we conclude that $(x, y) = \left(\frac{1}{2}, \frac{1}{12}\right)$ is on the trail of $P$. Since $C$ and $A$ cannot be congruent, $x_0 \ne 1, x \ne \frac{2}{3}$.

Therefore the equation of the trail is

$$y = \frac{1}{3}(3x-1)^2, \ x \ne \frac{2}{3}.$$

**2nd solution :**

From 1st solution, the equation of $AB$ is

$y = 2x - 1$, $B(0, -1)$, $D\left(\frac{1}{2}, 0\right)$.

Thus $D$ is the midpoint of $AB$.

Set $\gamma = \frac{CD}{CP}$, $t_1 = \frac{CA}{CE} = 1 + \lambda_1$, $t_2 = \frac{CB}{CF} = 1 + \lambda_2$.

Then $t_1 + t_2 = 3$.

Since $AD$ is a median of $\Delta ABC$, $S_{\Delta CAB} = 2S_{\Delta CAD} = 2S_{\Delta CBD}$ where $S_\Delta$ denotes the area of $\Delta$.

But $\frac{1}{t_1 t_2} = \frac{CE \cdot CF}{CA \cdot CB} = \frac{S_{\Delta CEF}}{S_{\Delta CAB}} = \frac{S_{\Delta CEP}}{2S_{\Delta CAD}} + \frac{S_{\Delta CFP}}{2S_{\Delta CED}}$

$$= \frac{1}{2}\left(\frac{1}{t_1\gamma} + \frac{1}{t_2\gamma}\right) = \frac{t_1+t_2}{2t_1t_2\gamma} = \frac{3}{2t_1t_2\gamma},$$

So $\gamma = \frac{3}{2}$ and $P$ is the center of gravity for $\Delta ABC$.

Consider $P(x, y)$ and $C(x_0, x_0^2)$. Since $C$ is different from $A$, $x_0 \ne 1$. Thus the coordinates of the center of gravity $P$ are $x = \frac{0+1+x_0}{3} = \frac{1+x_0}{3}$, $x \ne \frac{2}{3}$,

$y = \frac{-1+1+x_0^2}{3} = \frac{x_0^2}{3}$.

Eliminating $x_0$, we get $y = \frac{1}{3}(3x-1)^2$.

Thus the equation of the trail is

$$y = \frac{1}{3}(3x-1)^2, \ x \ne \frac{2}{3}.$$

**3.** (1) Let $\alpha \le x_1 < x_2 \le \beta$, then

$4x_1^2 - 4tx_1 - 1 \le 0$, $4x_2^2 - 4tx_2 - 1 \le 0$.

Therefore, $4(x_1^2 + x_2^2) - 4t(x_2 + x_2) - 2 \le 0$,

$2x_1x_2 - t(x_1 + x_2) - \frac{1}{2} < 0$.

But $f(x_2)-f(x_1)=\dfrac{2x_2-t}{x_2^2+1}-\dfrac{2x_1-t}{x_1^2+1}$

$$=\frac{(x_2-x_1)[t(x_1+x_2)-2x_1x_2+2]}{(x_2^2+1)\,(x_1^2+1)},$$

and $t(x_1 + x_2) - 2x_1x_2 + 2$

$> t(x_1 + x_2) - 2x_1x_2 + \dfrac{1}{2} > 0$, thus $f(x_2 - f(x_1) > 0$.

Consequently, $f(x)$ is an increasing function on the interval $[\alpha, \beta]$.

Since $\alpha + \beta = t$ and $\alpha\beta = -\dfrac{1}{4}$,

$g(t) = \max f(x) - \min f(x) = f(\beta) - f(\alpha)$

$$=\frac{\sqrt{t^2+1}\left(t^2+\frac{5}{2}\right)}{t^2+\frac{25}{16}}=\frac{8\sqrt{t^2+1}\,(2t^2+5)}{16t^2+25}$$

(2) $$g(\tan u_i)=\frac{\dfrac{8}{\cos u_i}\left(\dfrac{2}{\cos^2 u_i}+3\right)}{\dfrac{16}{\cos^2 u_i}+9}$$

$$=\frac{\dfrac{16}{\cos u_i}+24\cos u_i}{16+9\cos^2 u_i}$$

$$\geq\frac{2\sqrt{16\times 24}}{16+9\cos^2 u_i}=\frac{16\sqrt{6}}{16+9\cos^2 u_i}\quad (i=1,2,3)$$

So, $$\sum_{i=1}^{3}\frac{1}{g(\tan u_i)}\leq\frac{1}{16\sqrt{6}}\sum_{i=1}^{3}(16+9\cos^2 u_i)$$

$$=\frac{1}{16\sqrt{6}}\left(16\times 3+9\times 3-9\sum_{i=1}^{3}\sin^2 u_i\right)$$

Since $\sum_{i=1}^{3}\sin u_i=1$ and $u_i\in\left(0,\dfrac{\pi}{2}\right)$,

$i = 1, 2, 3$, we obtain

$$3\sum_{i=1}^{3}\sin^2 u_i>\left(\sum_{i=1}^{3}\sin u_i\right)^2=1.$$

Thus, $$\frac{1}{g(\tan u_1)}+\frac{1}{g(\tan u_2)}+\frac{1}{g(\tan u_3)}$$

$$<\frac{1}{16\sqrt{6}}\left(75-9\times\frac{1}{3}\right)=\frac{3}{4}\sqrt{6}.$$

**Remark :** Part (1) of this problem is well-known, we put in an inequality to increase the level of difficulty.

**4. 1st solution :**

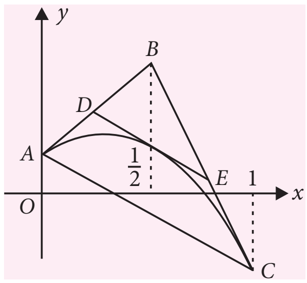
Let $z = x + yi$ $(x, y \in R)$, then

$$x+yi=a\cos^4 t\cdot i+2\left(\frac{1}{2}+bi\right)\cos^2 t\cdot\sin^2 t$$
$$+(1+ci)\sin^4 t.$$

Separating real and imaginary parts, we get

$x = \cos^2 t\cdot\sin^2 t + \sin^4 t = \sin^2 t$,

$y = a(1 - x)^2 + 2b(1 - x)x + cx^2$ $(0 \leq x \leq 1)$




That is,

$y = (a + c - 2b)x^2 + 2(b - a)x + a$ $(0 \leq x \leq 1)$ ...(i)

Since $A$, $B$, $C$ are non-collinear, $a + c - 2b \neq 0$.

So equation (i) is the segment of a parabola (see the diagram). Furthermore, the mid points of $AB$ and $BC$ are $D\left(\dfrac{1}{4},\dfrac{a+b}{2}\right)$ and $E\left(\dfrac{3}{4},\dfrac{b+c}{2}\right)$, respectively. So the equation of line $DE$ is

$$y=(c-a)x+\frac{1}{4}(3a+2b-c)\qquad ..(ii)$$

Solving equations (i) and (ii) simultaneously, we get $(a+c-2b)\left(x-\dfrac{1}{2}\right)^2=0$. Then $x=\dfrac{1}{2}$, since $a + c - 2b \neq 0$. So the parabola and line $DE$ have one and only one common point $P\left(\dfrac{1}{2},\dfrac{a+c+2b}{4}\right)$.

Notice that $\dfrac{1}{4}<\dfrac{1}{2}<\dfrac{3}{4}$, so point $P$ is on the segment $DE$ and satisfies equation (i), as required.

**2nd solution :**

We can solve the problem using the method of complex numbers directly. Let $D$, $E$ be the mid points of $AB$, $CB$, respectively.

Then the complex numbers corresponding to $D$, $E$ are $\dfrac{1}{2}(z_0+z_1)=\dfrac{1}{4}+\dfrac{a+b}{2}i$, $\dfrac{1}{2}(z_1+z_2)=\dfrac{3}{4}+\dfrac{b+c}{2}i$, respectively. So, complex

number $z$ corresponding to a point on the segment $DE$ satisfies

$$z = \lambda\left(\frac{1}{4}+\frac{a+b}{2}i\right)+(1-\lambda)\left(\frac{3}{4}+\frac{b+c}{2}i\right), 0 \le \lambda \le 1$$

Substitute the above expression into the equation of the curve $z = z_0\cos^4 t + 2z_1\cos^2 t \cdot \sin^2 t + z_2\sin^4 t$, and separate the real and imaginary parts from both sides to give the following two equations,

$$\begin{cases} \frac{3}{4}-\frac{\lambda}{2} = \sin^2 t\cos^2 t + \sin^4 t, \\ \frac{1}{2}[\lambda a + b(1-\lambda)c] = a\cos^4 t + 2b\sin^2 t\cos^2 t + c\sin^4 t \end{cases}$$

Eliminating $\lambda$ from the equations, we get

$$\frac{3}{4}(a-c)+\frac{b+c}{2}$$
$= a\cos^4 t + (2b + a - c)\sin^2 t\cos^2 t + a\sin^4 t$
$= a(1 - 2\sin^2 t\cos^2 t) + (2b + a - c)\sin^2 t\cos^2 t$
$= a + (2b - a - c)\sin^2 t\cos^2 t.$

Then $(2b - a - c)\left(\sin^2 t\cos^2 t - \frac{1}{4}\right) = 0$.

Since $A$, $B$, $C$ are non-collinear, we know that $z_1 \ne \frac{1}{2}(z_0+z_2)$. So $2b - a - c \ne 0$.

Then $\sin^2 t\cos^2 t = \sin^2 t(1 - \sin^2 t) = \frac{1}{4}$, that is,



$\left(\sin^2 t - \frac{1}{2}\right)^2 = 0$. Then we have $\frac{3}{4}-\frac{\lambda}{2} = \frac{1}{4}+\left(\frac{1}{2}\right)^2 = \frac{1}{2}$, so $\lambda = \frac{1}{2} \in [0, 1]$. That means that the curve and the line $DE$ have one and only one common point and the complex number corresponding to this common point is

$$z = \frac{1}{2}\left(\frac{1}{4}+\frac{a+b}{2}i\right)+\frac{1}{2}\left(\frac{3}{4}+\frac{b+c}{2}i\right)$$
$$= \frac{1}{2}+\frac{a+c+2b}{4}i$$

**5.** Working out the first few terms gives us an idea of how the given sequence develops:

| $n$ | $s_{2n-1}$ | $s_{2n}$ |
|---|---|---|
| 1 | $a$ | $b$ |
| 2 | $b - a$ | $2b - a$ |
| 3 | $b$ | $3b - a$ |
| 4 | $2b - a$ | $5b - 2a$ |
| 5 | $3b - a$ | $8b - 3a$ |
| 6 | $5b - 2a$ | $13b - 5a$ |
| 7 | $8b - 3a$ | $21b - 8a$ |

It appears that the coefficients in the even terms form a Fibonacci sequence and from the 5th term, every odd term is a repeat of the third term before it.

These observations are true for the entire sequence since, for $m \ge 1$, we have:

$s_{2m+2} = s_{2m+1} + s_{2m}$
$s_{2m+3} = s_{2m+2} - s_{2m+1} = s_{2m}$
$s_{2m+4} = s_{2m+3} + s_{2m+2} = s_{2m+2} + s_{2m}$

So, defining $F_1 = 1$, $F_2 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$, we have $s_{2n} = bF_n - aF_{n-2}$ for $n \ge 3$. Since $a = 3$ and $b < 1000$, none of the first five terms of the given sequence equal 2015. So we are looking for integer solutions of

$bF_n - 3F_{n-2} = 2015$ for $n \ge 3$.
$s_6 = 3b - 3 = 2015$, has no solution.
$s_8 = 5b - 6 = 2015$, has no solution.
$s_{10} = 8b - 9 = 2015$ implies $b = 253$.

For $n \ge 6$ we have $b = 2015/F_n + 3F_{n-2}/F_n$. Since $F_n$ increases, we have $F_n \ge 13$ and $F_{n-2}/F_n < 1$ for $n \ge 6$.

Hence $b < 2015/13 + 3 = 158$. So the largest value of $b$ is 253.

# WB JEE MOCK TEST PAPER

**Series-3**

**The entire syllabus of Mathematics of WB-JEE is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:**

| Unit No. | Topic | Syllabus In Details |
|---|---|---|
| UNIT NO. 3 | Permutations and Combinations | Fundamental principle of counting, permutation as an arrangement and combination as selection, meaning of $P(n, r)$ and $C(n, r)$, simple applications. |
| | Trigonometry | General solution and Properties of Triangle. |
| | Logarithms | Logarithm and their properties. |
| | Co-ordinate Geometry-2D | Circles: Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equations of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent. |

**Time : 1 hr 15 min** **Full marks : 50**

## CATEGORY-I

**For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of full mark (1/4) will be deducted. If candidate marks more than one answer, negative marking will be done.**

1. A person is permitted to select atleast one and at most $n$ coins from a collection of $2n + 1$ distinct coins. If the total number of ways in which he can select coins is 255, then $n$ equals
(a) 4 (b) 8 (c) 16 (d) 32

2. The letters of the word 'ARRANGE' is so arranged that two R's do not come together. Number of such words is
(a) 540 (b) 900
(c) 1080 (d) none of these

3. The number of ways in which 6 oranges of different sizes can be distributed among 6 boys of different ages so that the largest orange is always given to the youngest boy, is
(a) 720 (b) 600
(c) 120 (d) none of these

4. The value of $\sum_{r=1}^{n} \frac{{}^nP_r}{r!}$ will be
(a) $2^n$ (b) $2^n - 1$
(c) $2^n + 1$ (d) $2^{n-1}$

5. The value of ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to
(a) ${}^{47}C_5$ (b) ${}^{52}C_5$ (c) ${}^{52}C_4$ (d) ${}^{52}C_3$

6. The number of 5 digit numbers that can be formed with the digits 0, 1, 2, 3, 4, 5 which are divisible by 3 and no digit can be repeated in any number, is
(a) 216 (b) 120 (c) 240 (d) 312

7. There are 9 points in a plane of which no three are collinear and 4 points are concyclic. The number of different circles that can be drawn through atleast 3 points of these points is
(a) 80 (b) 81
(c) 84 (d) none of these

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**8.** Five speakers $A$, $B$, $C$, $D$ and $E$ have been asked to deliver a lecture in a meeting. In how many ways can their lectures be arranged so that $C$ delivers lecture just after $A$?
(a) 48 (b) 24
(c) 60 (d) none of these

**9.** The equation $\sqrt{3}\sin x + \cos x = 4$ has
(a) infinitely many solutions
(b) no solution (c) two solutions
(d) only one solution

**10.** If $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0$, when $0 < \theta < \pi$, then the values of $\theta$ are
(a) $\frac{\pi}{3} \pm \pi$ (b) $\frac{\pi}{3}, \cos^{-1}\left(\frac{3}{5}\right)$
(c) $\cos^{-1}\left(\frac{3}{5}\right) \pm \pi$ (d) $\frac{\pi}{3}, \pi - \cos^{-1}\left(\frac{3}{5}\right)$

**11.** The most general solution of the equation $\log_{\cos\theta}\tan\theta + \log_{\sin\theta}\cot\theta = 0$, is
(a) $n\pi + \frac{\pi}{4}, n \in Z$ (b) $n\pi - \frac{\pi}{4}, n \in Z$
(c) $2n\pi - \frac{\pi}{4}, n \in Z$ (d) $2n\pi + \frac{\pi}{4}, n \in Z$

**12.** The general solution of $2 - \cos x = 2\tan\frac{x}{2}$ is
(a) $(2n+1)\frac{\pi}{2}, n \in Z$ (b) $(4n+1)\frac{\pi}{2}, n \in Z$
(c) $2n\pi, n \in Z$ (d) $(4n + 1)\pi, n \in Z$

**13.** The general solution of
$$\sin x + \cos x = \min_{a \in R}\{1, a^2 - 4a + 6\} \text{ is}$$
(a) $\frac{n\pi}{2} + (-1)^n\frac{\pi}{4}, n \in Z$ (b) $2n\pi + (-1)^n\frac{\pi}{4}, n \in Z$
(c) $n\pi + (-1)^{n+1}\frac{\pi}{4}, n \in Z$
(d) $n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}, n \in Z$

**14.** The equation $\cos^4\theta + \sin^4\theta + \lambda = 0$ has real solution for $\theta$, if
(a) $\frac{3}{4} \le \lambda \le 1$ (b) $-1 \le \lambda \le -\frac{1}{2}$
(c) $0 \le \lambda \le 1$ (d) $\lambda < -1$

**15.** In a triangle $ABC$, if $\angle B = \frac{\pi}{3}$, $\angle C = \frac{\pi}{4}$ and $D$ divides $BC$ internally in the ratio 1 : 3, then $\frac{\sin\angle BAD}{\sin\angle CAD}$ equals
(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{6}}$ (d) $\sqrt{\frac{2}{3}}$

**16.** The ratio of the sides of a triangle is 4 : 5 : 7, then the triangle must be
(a) right-angled
(b) acute-angled
(c) obtuse-angled
(d) right-angled and isosceles

**17.** In a triangle $ABC$, $(a + b + c)(b + c - a) = \lambda bc$, if
(a) $\lambda < 0$ (b) $0 \le \lambda \le 4$
(c) $0 \le \lambda < 4$ (d) $0 < \lambda \le 4$

**18.** If two sides of a triangle are $2\sqrt{3} - 2$ and $2\sqrt{3} + 2$ and their included angle is 60°, then the other angles are
(a) 75°, 45° (b) 105°, 15°
(c) 60°, 60° (d) 90°, 30°

**19.** In a triangle $ABC$, if $B = \frac{\pi}{4}, C = \frac{\pi}{3}$ and $a = (\sqrt{3} + 1)$ cm, then the area of the triangle is
(a) $\frac{\sqrt{3}+1}{2}\text{cm}^2$ (b) $\frac{3+\sqrt{3}}{2}\text{cm}^2$
(c) $\frac{\sqrt{3}}{2}\text{cm}^2$ (d) $\frac{\sqrt{3}-1}{2}\text{cm}^2$

**20.** Two adjacent sides of a cyclic quadrilateral are 3 and 5 and the angle between them is 60°. If the third side is 2, then the remaining fourth side is
(a) 2 (b) 3 (c) 5 (d) 4

**21.** The circles $x^2 + y^2 + 6x + 6y = 0$ and $x^2 + y^2 - 12x - 12y = 0$
(a) cut orthogonally
(b) touch each other internally
(c) intersect at two points
(d) touch each other externally

**22.** If the ends of the diameter of a circle are the points (0, 0) and $\left(a^3, \frac{1}{a^3}\right)$, then through which of the following points the circle passes?
(a) $\left(a, \frac{1}{a}\right)$ (b) $\left(a^2, \frac{1}{a^2}\right)$
(c) $\left(\frac{1}{a^2}, a^2\right)$ (d) $\left(\frac{1}{a}, a\right)$

23. If the circles $x^2 + y^2 - 4rx - 2ry + 4r^2 = 0$ and $x^2 + y^2 = 25$ touch each other, then $r$ satisfies
(a) $4r^2 + 10r \pm 25 = 0$
(b) $5r^2 + 10r \pm 16 = 0$
(c) $4r^2 \pm 10r + 25 = 0$
(d) $4r^2 \pm 10r - 25 = 0$

24. If the points (0, 0), (1, 0), (0, –1) and $(\lambda, 3\lambda)$ are concyclic, then $\lambda$ is
(a) 5 (b) 1/5 (c) –5 (d) –1/5

25. The shortest distance of the point (9, –12) from the circle $x^2 + y^2 = 16$, is
(a) 7 units (b) 11 units
(c) 15 units (d) 4 units

26. One extremity of a diameter of the circle $x^2 + y^2 - 8x - 4y + 15 = 0$ is (2, 1), the other extremity is
(a) (0, 0) (b) (6, 3)
(c) (4, 2) (d) (–3, –6)

27. The triangle $PQR$ is inscribed in the circle $x^2 + y^2 = 25$. If $Q$ and $R$ have coordinates (3, 4) and (–4, 3) respectively, then $\angle QPR$ is equal to
(a) $\pi/2$ (b) $\pi/3$ (c) $\pi/4$ (d) $\pi/6$

28. The equation of a circle which passes through the point $(h, k)$ and touches the $y$-axis at origin, is
(a) $h^2(x^2 + y^2) = (h^2 + k^2)x$
(b) $h^2(x^2 + y^2) = (h^2 + k^2)y$
(c) $k^2(x^2 + y^2) = (h^2 + k^2)x$
(d) $k^2(x^2 + y^2) = (h^2 + k^2)y$

29. If $ax^2 + (2a - 3)y^2 - 6x + ay - 3 = 0$ represents a circle, then its radius is
(a) 1 (b) $\sqrt{6}$ (c) 1/2 (d) 3/2

30. Locus of a point which divides chord at a distance 1 unit from the centre of the circle $x^2 + y^2 = 1$ in the ratio 2 : 1 is
(a) $x^2 + y^2 = 2$ (b) $x^2 + y^2 = 4$
(c) $x^2 + y^2 = 8$ (d) $x^2 + y^2 = 16$

## CATEGORY-II

**Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidate marks more than one answer, negative marking will be done.**

31. If intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is $AB$, then equation of the circle with $AB$ as diameter is
(a) $x^2 + y^2 + x + y = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 - x - y = 0$
(d) $x^2 + y^2 + x - y = 0$

32. The equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter is
(a) $x^2 + y^2 + 3x + y + 11 = 0$
(b) $x^2 + y^2 - 3x - y - 11 = 0$
(c) $x^2 + y^2 + 3x + y - 11 = 0$
(d) $x^2 + y^2 + 3x - y - 11 = 0$

33. In a triangle $ABC$, if $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, then the sides $a, b, c$
(a) satisfy $a + b = c$ (b) are in A.P.
(c) are in G.P. (d) are in H.P.

34. The number of integral values of $k$ for which the equation $3\cos x + 4\sin x = 2k + 1$ has a solution, is
(a) 3 (b) 6 (c) 4 (d) 5

35. The number of ways in which the letters of the word 'COMBINE' can be arranged so that the word begin and end with a vowel, is
(a) 30 (b) 504 (c) 360 (d) 720

## CATEGORY-III

**In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If candidate marks one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without making any incorrect, formula below will be used to allot marks.**

**2×(no. of correct response/total no. of correct options)**

36. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then $n$ is
(a) 8 (b) 9 (c) 14 (d) 7

37. If $0 \le x \le 2\pi$ and $|\cos x| \le \sin x$, then
(a) the set of values of $x$ is $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(b) the number of solutions that are integral multiples of $\pi/4$ is three
(c) the sum of the largest and the smallest solution is $3\pi/4$
(d) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$

38. In a $\Delta ABC$, $\tan A$ and $\tan B$ are the roots of the equation $ab(x^2 + 1) = c^2x$, where $a$, $b$ and $c$ are the sides of the triangle. Then
(a) $\tan(A - B) = \frac{a^2 - b^2}{2ab}$
(b) $\cot C = 0$
(c) $\sin^2 A + \sin^2 B = 1$
(d) none of these

**39.** In a triangle $ABC$, $2\cos\frac{A-C}{2} = \frac{a+c}{\sqrt{a^2+c^2-ac}}$. Then

(a) $B = \pi/3$ (b) $B = C$

(c) $A, B, C$ are in A.P.

(d) $B + C = A$

**40.** A line parallel to the line $x - 3y = 2$ touches the circle $x^2 + y^2 - 4x + 2y - 5 = 0$ at the point

(a) $(1, -4)$ (b) $(1, 2)$

(c) $(3, -4)$ (d) $(3, 2)$

## SOLUTIONS

**1. (a) :** The total number of ways, the person can select $n$ coins from $2n + 1$ coins is

$$^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = 255$$

$$\Rightarrow 1 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = 255 + 1$$

$$\Rightarrow \frac{1}{2}(2^{2n+1}) = 256 \Rightarrow 2^{2n} = 2^8 \Rightarrow 2n = 8 \Rightarrow n = 4.$$

**2. (b) :** The word ARRANGE consist of 7 letters. Among these 7 letters there are two R's and two A's. No. of ways of arranging the letters such that two R's are always together $= \frac{6!}{2!}$

$\therefore$ The numbers of words where two R's do not come together are $\frac{7!}{2!2!} - \frac{6!}{2!} = \frac{6!}{2!}\left(\frac{7}{2} - 1\right) = \frac{6!}{2!} \cdot \frac{5}{2} = 900.$

**3. (c):** After giving the largest fruit to the youngest boy, the remaining 5 fruits can be given to the remaining 5 boys in 5! ways, *i.e.* in 120 ways.

**4. (b) :** We know that $\frac{^nP_r}{r!} = {}^nC_r \Rightarrow \sum_{r=1}^{n} \frac{^nP_r}{r!} = \sum_{r=1}^{n} {}^nC_r$

Now $\sum_{r=1}^{n} {}^nC_r = {}^nC_0 + ({}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n) - {}^nC_0$

$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n) - {}^nC_0 = 2^n - 1$$

**5. (c):** We have $^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

$$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$$

$$= ({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$

$$= ({}^{48}C_4 + {}^{48}C_3) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$

$$= ({}^{49}C_4 + {}^{49}C_3) + {}^{50}C_3 + {}^{51}C_3$$

$$= ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3 = ({}^{51}C_4 + {}^{51}C_3) = {}^{52}C_4$$

**6. (a) :** The required number of numbers are $5! + (5! - 4!) = 120 + 96 = 216$

**7. (b) :** To form a circle atleast 3 points are required.

$\therefore$ Number of circles which can be drawn by 9 points $= {}^9C_3$, but given that 4 points are concyclic.

Therefore, instead of getting $^4C_3$ number of circles we get only one circle.

Hence total number of required circles is

$$^9C_3 - {}^4C_3 + 1 = 81$$

**8. (b) :** The arrangement of lectures of the 5 persons may be done as $(AC)BDE$, where $C$ always deliver lecture after $A$. Thus total number of such arrangements are $4! = 24$

**9. (b) :** We have $\sqrt{3}\sin x + \cos x = 4$

$$\Rightarrow \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2$$

$$\Rightarrow \sin x \sin\frac{\pi}{3} + \cos x \cos\frac{\pi}{3} = 2$$

$\Rightarrow \cos\left(x - \frac{\pi}{3}\right) = 2$, which is impossible.

As we know that $-1 \le \cos\left(x - \frac{\pi}{3}\right) \le 1$.

**10. (d) :** We have $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0, \quad 0 < \theta < \pi$

$\Rightarrow 5\cos2\theta + 1 + \cos\theta + 1 = 0$

$\Rightarrow 5(2\cos^2\theta - 1) + \cos\theta + 2 = 0$

$\Rightarrow 10\cos^2\theta + \cos\theta - 3 = 0$

$\Rightarrow (2\cos\theta - 1)(5\cos\theta + 3) = 0$

$$\therefore \theta = \frac{\pi}{3} \text{ and } \theta = \cos^{-1}\left(-\frac{3}{5}\right) = \pi - \cos^{-1}\left(\frac{3}{5}\right)$$

$(\because 0 < \theta < \pi)$

**11. (a) :** We have $\log_{\cos\theta}\tan\theta + \log_{\sin\theta}\cot\theta = 0$

$$\Rightarrow \log_{\cos\theta}\tan\theta - \log_{\sin\theta}\tan\theta = 0$$

$$\Rightarrow \frac{\log\tan\theta}{\log\cos\theta} = \frac{\log\tan\theta}{\log\sin\theta}$$

$$\Rightarrow \frac{\log\sin\theta}{\log\cos\theta} = 1$$

$$\Rightarrow \log_{\cos\theta}\sin\theta = 1 \Rightarrow \cos\theta = \sin\theta \Rightarrow \tan\theta = \tan\frac{\pi}{4}$$

$$\therefore \theta = n\pi + \frac{\pi}{4}, \ n \in Z.$$

**12. (b) :** We have

$$2-\cos x = 2\tan\frac{x}{2} \Rightarrow 2\left(1-\tan\frac{x}{2}\right)=\cos x$$

$$\Rightarrow 2\left(1-\tan\frac{x}{2}\right)=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\Rightarrow \left(1-\tan\frac{x}{2}\right)\left(2-\frac{1+\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)=0$$

Either $1-\tan\frac{x}{2}=0$

$$\therefore \tan\frac{x}{2}=1=\tan\frac{\pi}{4} \Rightarrow \frac{x}{2}=n\pi+\frac{\pi}{4},\ n\in Z$$

$$\Rightarrow x=2n\pi+\frac{\pi}{2},\ n\in z \Rightarrow x=(4n+1)\frac{\pi}{2},\ n\in Z$$

or $2-\dfrac{1+\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=0 \Rightarrow 2\tan^2\frac{x}{2}-\tan\frac{x}{2}+1=0$

$$\Rightarrow \tan\frac{x}{2}=\frac{1\pm\sqrt{1^2-4\cdot 2\cdot 1}}{2\cdot 2}=\frac{1\pm i\sqrt{7}}{4}$$

$\Rightarrow$ no solution

**13. (d) :** Here $a^2 - 4a + 6 = (a - 2)^2 + 2 \geq 2$

$$\therefore \min_{a\in R}\{1, a^2-4a+6\}=1$$

Now, $\sin x+\cos x=1 \Rightarrow \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{\sqrt{2}}$

$$\Rightarrow \sin\left(x+\frac{\pi}{4}\right)=\sin\frac{\pi}{4} \Rightarrow x+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$$

$$\Rightarrow x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4},\ n\in Z$$

**14. (b) :** We have $1 - 2\sin^2\theta\cos^2\theta = -\lambda$

$$\Rightarrow 1-\frac{1}{2}\sin^2 2\theta=-\lambda$$

$$\Rightarrow 1-\frac{1}{4}(1-\cos 4\theta)=-\lambda \Rightarrow \frac{3}{4}+\frac{1}{4}\cos 4\theta=-\lambda$$

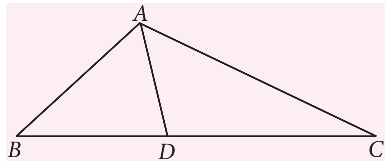
$\because\ -1\leq\cos 4\theta\leq 1$

$$\therefore -\frac{1}{4}\leq\frac{1}{4}\cos 4\theta\leq\frac{1}{4} \Rightarrow \frac{3}{4}-\frac{1}{4}\leq\frac{3}{4}+\frac{1}{4}\cos 4\theta\leq\frac{3}{4}+\frac{1}{4}$$

$$\Rightarrow \frac{1}{2}\leq-\lambda\leq 1 \Rightarrow -1\leq\lambda\leq-\frac{1}{2}$$

**15. (c):** From $\Delta ABD$, we have $\dfrac{BD}{\sin\angle BAD}=\dfrac{AD}{\sin B}$

From $\Delta ACD$, we have $\dfrac{CD}{\sin\angle CAD}=\dfrac{AD}{\sin C}$




$\because\ BD : CD = 1 : 3$

$$\therefore \frac{AD\sin\angle BAD}{\sin B}:\frac{AD\sin\angle CAD}{\sin C}=1:3$$

$$\Rightarrow \frac{\sin\angle BAD}{\sin\frac{\pi}{3}}:\frac{\sin\angle CAD}{\sin\frac{\pi}{4}}=1:3 \Rightarrow \frac{\sqrt{2}\sin\angle BAD}{\sqrt{3}\sin\angle CAD}=\frac{1}{3}$$

$$\Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}=\frac{1}{\sqrt{6}}$$

**16. (c):** Let the sides of the triangle $ABC$ be $4x$, $5x$ and $7x$

$$\therefore \cos A=\frac{(5x)^2+(4x)^2-(7x)^2}{2\cdot 5x\cdot 4x}=-\frac{1}{5}$$

$\Rightarrow$ the angle $A$ is an obtuse angle

Thus, $\Delta ABC$ is obtuse-angled

**17. (b) :** We have $(a + b + c)(b + c - a) = \lambda bc$

$\Rightarrow (b + c)^2 - a^2 = \lambda bc \Rightarrow b^2 + c^2 - a^2 = (\lambda - 2)bc$

$$\Rightarrow \frac{b^2+c^2-a^2}{2bc}=\frac{\lambda-2}{2} \Rightarrow \cos A=\frac{\lambda-2}{2}$$

$\because\ -1\leq\cos A\leq 1$

$$\therefore -1\leq\frac{\lambda-2}{2}\leq 1 \Rightarrow -2\leq\lambda-2\leq 2 \Rightarrow 0\leq\lambda\leq 4$$

**18. (b) :** Let $b=2\sqrt{3}+2,\ c=2\sqrt{3}-2$ and $A=60°$

$$\therefore \tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{4}{4\sqrt{3}}\cot 30°=1=\tan 45°$$

$\Rightarrow B - C = 90°$. Again, $B + C = 120°$ ($\because A = 60°$)

Therefore the other two angles are $B = 105°$ and $C = 15°$.

**19. (b) :** Here $A=\pi-\left(\frac{\pi}{4}+\frac{\pi}{3}\right)=\frac{5\pi}{12}=75°$.

Now $\dfrac{a}{\sin A}=\dfrac{b}{\sin B} \Rightarrow \dfrac{\sqrt{3}+1}{\sin 75°}=\dfrac{b}{\sin 45°}$

$$\therefore b=\frac{\sin 45°}{\sin 75°}(\sqrt{3}+1)=2$$

Area of the triangle $=\frac{1}{2}ab\sin C$

$$=\left\{\frac{1}{2}(\sqrt{3}+1)\cdot 2\sin\frac{\pi}{3}\right\}\text{ cm}^2=\left(\frac{3+\sqrt{3}}{2}\right)\text{cm}^2$$

**20. (b) :** Let $AB = 3$, $BC = 5$, $CD = 2$, $DA = x$ and $\angle ABC = 60°$ of the cyclic quadrilateral $ABCD$

$\therefore\ \angle CDA = 180° - 60° = 120°$

From $\Delta ABC$, we have $\cos\angle ABC=\frac{AB^2+BC^2-AC^2}{2AB\cdot BC}$

$\Rightarrow \frac{1}{2}=\frac{9+25-AC^2}{2\cdot 3\cdot 5} \Rightarrow AC^2 = 19$ ... (1)

From $\Delta ADC$, we have $\cos\angle CDA=\frac{CD^2+AD^2-AC^2}{2CD\cdot AD}$

$\Rightarrow \frac{-1}{2}=\frac{4+x^2-19}{2\cdot x\cdot 2}$ (Using (1))

$\Rightarrow x^2 + 2x - 15 = 0 \Rightarrow (x + 5)(x - 3) = 0$

$\Rightarrow x = 3\ (\because x \neq -5)$

**21. (d) :** For the circle, $x^2 + y^2 + 6x + 6y = 0$, the centre is at $(-3, -3)$ and radius $= 3\sqrt{2}$

Also the centre and radius of the circle $x^2 + y^2 - 12x - 12y = 0$ is $(6, 6)$ and $6\sqrt{2}$ respectively.

Distance between the centres of the circles

$$=\sqrt{81+81}=9\sqrt{2}=\text{sum of the radii}$$

$\therefore$ the two circles touch each other externally.

**22. (d) :** The equation of the circle whose diameter's end points are $(0, 0)$ and $\left(a^3,\frac{1}{a^3}\right)$ is

$$(x-0)(x-a^3)+(y-0)\left(y-\frac{1}{a^3}\right)=0 \quad .... (1)$$

Since the equation (1) is satisfied by $\left(\frac{1}{a},a\right)$, therefore the circle is passing through $\left(\frac{1}{a},a\right)$.

**23. (d) : Hint:** Two circles touch each other if the distance between their centres is equal to the sum or difference of the their radii.

**24. (d) :** Equation of the circle passing through the points $(0, 0)$, $(1, 0)$, $(0, -1)$ is

$\Rightarrow x^2 + y^2 - x + y = 0$ ... (1)

$(0, 0)$, $(1, 0)$, $(0, -1)$ and $(\lambda, 3\lambda)$ are concyclic, therefore from (1), we get

$\lambda^2 + 9\lambda^2 - \lambda + 3\lambda = 0 \Rightarrow 10\lambda^2 + 2\lambda = 0$

$\Rightarrow 2\lambda(5\lambda + 1) = 0 \Rightarrow \lambda=-\frac{1}{5}\ (\because \lambda \neq 0)$

**25. (b) :** The centre and radius of the circle $x^2 + y^2 = 16$ are respectively $(0, 0)$ and 4.

Now the distance of the point $(9, -12)$ from $(0, 0)$ is

$$\sqrt{81+144}=\sqrt{225}=15.$$

Hence, the least distance of the circle $x^2 + y^2 = 16$ from the point $(9, -12)$ is $15 - 4 = 11$ units

**26. (b) :** Here, the given equation of the circle is

$x^2 + y^2 - 8x - 4y + 15 = 0$

$\therefore$ Centre is $(4, 2)$. Given that one extremity of the diameter is $(2, 1)$. Let the other extremity be $(x, y)$.

Therefore the other extremity of the diameter is $(6, 3)$

[Since the centre is the mid-point of diameter]

**27. (c):** The centre of the circle $x^2 + y^2 = 25$ is $(0, 0)$.

Now slope of $QO \times$ slope of $RO$

$$=\frac{4-0}{3-0}\times\frac{3-0}{-4-0}=\frac{4}{3}\times-\frac{3}{4}=-1$$

Therefore angle at the centre $(\angle QOR)=\frac{\pi}{2}$,

$\therefore$ Angle at the circumference is

$$\angle QPR=\frac{1}{2}\times\angle QOR=\frac{\pi}{4}$$

**28. (a) :** Let the equation of the circle be

$(x - \alpha)^2 + (y)^2 = \alpha^2$

($\because$ the circle touches $y$-axis at origin. $\therefore$ centre is on $x$-axis)

$\Rightarrow x^2 - 2\alpha x + \alpha^2 + y^2 = \alpha^2$

$\Rightarrow x^2 - 2\alpha x + y^2 = 0$ ... (1)

Since (1) is passing through $(h, k)$, therefore

$$\alpha=\frac{h^2+k^2}{2h}$$

Now (1) becomes

$$x^2+y^2=2\left(\frac{h^2+k^2}{2h}\right)x \Rightarrow h(x^2+y^2)=(h^2+k^2)x$$

**29. (d) :** The given equation is

$ax^2 + (2a - 3)y^2 - 6x + ay - 3 = 0$

This equation will represent a circle if the coefficient of $x^2$ is equal to the coefficient of $y^2$.

$\therefore\ a = 2a - 3 \Rightarrow a = 3.$

Now, the equation of the circle is

$3x^2 + 3y^2 - 6x + 3y - 3 = 0$

$\Rightarrow x^2 + y^2 - 2x + y - 1 = 0$

$\therefore$ Radius $=\sqrt{1+\frac{1}{4}+1}=\frac{3}{2}.$

**30. (a) :** Let $P(h, k)$ divides the chord $\overline{AB}$ in the ratio $2 : 1$. $\overline{OD} \perp \overline{AB}$, where $O$ is the centre of the circle and $D$ is the foot of the perpendicular drawn from the centre $O$.

Now, $\overline{OB} = \sqrt{10}$; $\overline{OD} = 1$.

$\therefore \ \overline{DB} = \sqrt{10-1} = 3 \quad \therefore \ \overline{AB} = 2;\ \overline{DB} = 6.$

$\because \ \overline{AP} : \overline{PB} = 2:1 \ \therefore \ \overline{PB} = \frac{1}{3}\overline{AB} = 2$

$\therefore \ \overline{DP} = \overline{DB} - \overline{PB} = 3 - 2 = 1.$

Now, from $\Delta ODP$, we have

$OP^2 = OD^2 + DP^2 \ \Rightarrow \ h^2 + k^2 = 1 + 1 = 2$

$\therefore$ locus of the point $(h, k)$ is $x^2 + y^2 = 2$

**31. (c):** The line $y = x$ cuts the circle $x^2 + y^2 - 2x = 0$ at the points $A(0, 0)$ and $B(1, 1)$. Therefore the equation of the circle, whose diameter is $AB$, is

$(x - 0)(x - 1) + (y - 0)(y - 1) = 0$

$\Rightarrow \ x^2 + y^2 - x - y = 0$

**32. (c):** Let the line $3x + y + 5 = 0$ cut the circle $x^2 + y^2 = 16$ at the points $A$ and $B$, hence $\overline{AB}$ is a chord of the circle.

The mid-point of the chord $\overline{AB}$ is $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ (the foot of the perpendicular from the centre to the chord) and length of the chord $\overline{AB} = 2\times\sqrt{16 - \frac{25}{10}} = 2\sqrt{\frac{27}{2}}$.

$\therefore$ Centre of the circle, whose diameter is $AB$, is

$\left(-\frac{3}{2}, -\frac{1}{2}\right)$ and its radius $= \frac{1}{2}AB = \sqrt{\frac{27}{2}}$.

Thus the equation of the circle is

$$\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{27}{2}$$

$\Rightarrow \ x^2 + y^2 + 3x + y = 11$

**33. (b) :** Given that, $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$

$\Rightarrow \ a(1 + \cos C) + c(1 + \cos A) = 3b$

$\Rightarrow \ a + c + (a\cos C + c\cos A) = 3b$

$\Rightarrow \ a + c + b = 3b \quad (\because a\cos C + c\cos A = b)$

$\Rightarrow \ a + c = 2b$, *i.e.* $a, b, c$ are in A.P.

**34. (b) :** We have $3\cos x + 4\sin x = 2k + 1$

$$\Rightarrow \ \frac{3}{\sqrt{3^2+4^2}}\cos x + \frac{4}{\sqrt{3^2+4^2}}\sin x = \frac{2k+1}{\sqrt{3^2+4^2}}$$

$$\Rightarrow \ \frac{3}{5}\cos x + \frac{4}{5}\sin x = \frac{2k+1}{5}$$

$\Rightarrow \ \cos(x - \alpha) = \frac{2k+1}{5}$, where $\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}$

$\because \ -1 \le \cos(x-\alpha) \le 1 \ \Rightarrow \ -1 \le \frac{2k+1}{5} \le 1$

$\Rightarrow \ -5 \le 2k + 1 \le 5 \ \Rightarrow \ -6 \le 2k \le 4$

Therefore integral values of $k$ are $-3, -2, -1, 0, 1, 2$

$\therefore$ Required number is 6.

**35. (d) :** The word 'COMBINE' is consisting of 7 letters. There are 3 vowels namely- E, I and O.

The two vowels for beginning and ending place can be arranged in ${}^3P_2$ ways. For each of the ${}^3P_2$ ways the remaining 5 letters can be arranged in 5! ways.

Therefore required number is ${}^3P_2 \times 5! = 720$.

**36. (c, d) :** Given that ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P.

$$\therefore \ 2\cdot\frac{n!}{5!(n-5)!} = \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!}$$

$$\Rightarrow \ \frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{6\times5}$$

$$\Rightarrow \ \frac{12}{(n-5)} = \frac{(n-4)(n-5)+30}{(n-4)(n-5)}$$

$\Rightarrow \ n^2 - 9n + 50 = 12(n - 4) \Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow \ n^2 - 14n - 7n + 98 = 0 \Rightarrow (n - 14)(n - 7) = 0$

$\Rightarrow \ n = 14$ or $n = 7$

**37. (b, d) :** We know that $|\cos x| \ge 0$

$\Rightarrow \ \sin x \ge 0$. So, there is no $x$ in $(\pi, 2\pi)$.

Now, if $x = 2\pi$, $|\cos 2\pi| \le \sin 2\pi$, which is not true.

So, $0 \le x \le \pi$

If $0 \le x \le \frac{\pi}{2}$ then $|\cos x| \le \sin x \Rightarrow \cos x \le \sin x$

$\Rightarrow \ \tan x \ge 1 \quad \therefore \ \frac{\pi}{4} \le x \le \frac{\pi}{2}$

If $\frac{\pi}{2} \le x \le \pi$ then $|\cos x| \le \sin x$

$\Rightarrow \ -\cos x \le \sin x \ \Rightarrow \ \tan x \le -1 \ \ (\because \cos x < 0)$

$\therefore \ \frac{\pi}{2} < x \le \frac{3\pi}{4} \ \therefore \ x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right].$

**Solution Sender of Maths Musing**

**SET-164**

| | | |
|---|---|---|
| 1. | Akash Banerjee | W.B. |
| 2. | Sk. Izajur Rahaman | W.B. |
| 3. | N. Jayanthi | Hyderabad |
| 4. | Khokon Kumar Nandi | W.B. |

**38. (a, b, c) :** Here $\tan A$, $\tan B$ are the roots of the equation $abx^2 - c^2x + ab = 0$

Therefore, $\tan A + \tan B = \frac{c^2}{ab}$ and $\tan A \tan B = 1$

Now $\tan A \tan B = 1 \Rightarrow \cos(A + B) = 0$
(as $A$, $B$ are both acute)

$\therefore \quad A + B = \frac{\pi}{2}$ *i.e.* $C = \frac{\pi}{2}$

Also $\cot C = \cot \frac{\pi}{2} = 0$

So $\Delta ABC$ is right angled at $C$ and

$\sin A = \cos B \Rightarrow \sin^2 A + \sin^2 B = 1$

Also $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{a}{b} - \frac{b}{a}}{1+1} = \frac{a^2 - b^2}{2ab}$.

**39. (a, c)**

**40. (b, c) :** Let $x - 3y + k = 0$ touch the circle at $(x_1, y_1)$
Then $xx_1 + yy_1 - 2(x + x_1) + y + y_1 - 5 = 0$
and $x - 3y + k = 0$ are identical

Hence, $\frac{x_1 - 2}{1} = \frac{y_1 + 1}{-3} = \frac{-2x_1 + y_1 - 5}{k}$

$$= \frac{2(x_1 - 2) - 1(y_1 + 1) + (-2x_1 + y_1 - 5)}{2 \times 1 - 1 \times (-3) + k}$$

$\therefore \quad \frac{x_1 - 2}{1} = \frac{y_1 + 1}{-3} = \frac{-10}{5+k} \Rightarrow x_1 = \frac{-10}{5+k} + 2 = \frac{2k}{5+k}$

and $y_1 = \frac{30}{5+k} - 1 = \frac{25-k}{5+k}$

$(x_1, y_1)$ is on the circle.

So, $\left(\frac{2k}{5+k}\right)^2 + \left(\frac{25-k}{5+k}\right)^2 - 4 \cdot \left(\frac{2k}{5+k}\right) + 2 \cdot \left(\frac{25-k}{5+k}\right) - 5 = 0$

On simplification, $k^2 + 10k - 75 = 0 \Rightarrow k = 5, -15$

$\therefore \; x_1 = \frac{10}{10} = 1,\; y_1 = \frac{20}{10} = 2$ or $x_1 = \frac{-30}{-10} = 3,\; y = \frac{25+15}{5-15} = -4.$

So, $(x_1, y_1) = (1, 2), (3, -4)$.

# Declining IIT cutoffs spark fears of falling standards

The IITs are struggling to get the best students, so much so that cutoff marks have been lowered in the past two years to fill seats in the country's premier engineering institutes.

Marks as low as 75 out of 372 in the joint entrance examination (JEE-Advance) this year were good enough to land a general category student a seat in one of the 23 IITs. Reserved category students —Dalits, tribals, and people with disability — had to get only 38 to secure admission.

This comes at a time the IIT Council decided to increase the number of students from around 70,000 to 100,000 over the next three years, enrolling even day scholars for the first time.

The lowered benchmark has raised questions about academic quality.

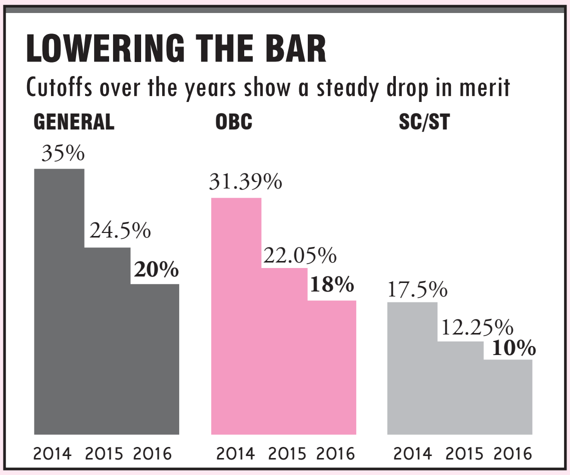
IIT-Roorkee expelled 11 BTech students early this year for poor academic performance. Last year, the premier institute expelled 72 students for dismal academic record, but re-admitted them later.

Experts said tough question papers and negative marking in the entrance examination had resulted in a shortfall of candidates meeting the benchmark, thus forcing the IIT management to lower the bar.

Doubts over a fall in standard have been dismissed.

"The teaching-learning process at IITs is intense, and many students who come with low ranks in JEE turn out to be gems when they pass the BTech programme," said the director of an IIT who didn't wish to be named.

A former director suggested improving the education standard at schools so that students could pass the entrance exam for IITs with higher grades. Of 147,678 students who appeared for this year's JEE (Advanced), 36,566 qualified.




"Maintaining quality is a major issue which is why IITs coach such students so that they are on par with the rest. Students who were expelled by IIT-Roorkee in 2015 have done much better now with extra help, coaching from other students and faculty."

*Courtesy : Hindustan Times*

# MATH archives


Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

**1.** The value of $\tan^2\frac{\pi}{16}+\tan^2\frac{2\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{6\pi}{16}+\tan^2\frac{7\pi}{16}$ is equal to

(a) 24 (b) 34
(c) 44 (d) none of these

**2.** If $f(x)=e^{\sin(x-[x])\cos\pi x}$, then $f(x)$ is ($[x]$ denotes the greatest integer function)
(a) non-periodic
(b) periodic with no fundamental period
(c) periodic with period 2
(d) periodic with period $\pi$

**3.** Which of the following homogeneous functions are of degree zero?

(a) $\frac{x}{y}\ln\frac{y}{x}+\frac{y}{x}\ln\frac{x}{y}$ (b) $\frac{x(x-y)}{y(x+y)}$

(c) $\frac{xy}{x^2+y^2}$ (d) all of these

**4.** Match the following.

| Column-I | | Column-II | |
|---|---|---|---|
| A. | $\lim\limits_{x\to0}\frac{\ln(\cos x)}{x}$ | P. | $-1$ |
| B. | $\lim\limits_{x\to1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$ | Q. | $-\frac{1}{2}$ |
| C. | $\lim\limits_{x\to0}\frac{x-\sin x}{x-\tan x}$ | R. | $0$ |

(a) A → P, B → R, C → Q
(b) A → Q, B → P, C → R
(c) A → R, B → P, C → Q
(d) none of these

**5.** If θ is small and positive number, then which of the following is/are correct?

(a) $\frac{\sin\theta}{\theta}=1$ (b) $\theta<\sin\theta<\tan\theta$

(c) $\frac{\tan\theta}{\theta}>\frac{\sin\theta}{\theta}$ (d) none of these

**6.** If $y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$, then prove that $\frac{dy}{dx}=\sqrt{\frac{3}{\cos x\cos 3x}}$.

**7.** Let $a\in R$, then prove that a function $f:R\to R$ is differentiable at $a$ if a function $\phi:R\to R$ satisfies $f(x)-f(a)=\phi(x)(x-a)\ \forall\, x\in R$ and $\phi$ is continuous at '$a$'.

**8.** If $\beta,\gamma\in(0,\pi)$ such that $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta+\gamma)=0$ and $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+\beta+\gamma)=0$. Then evaluate $f'(\beta)$ and $\lim\limits_{x\to\gamma}g(x)$, where $f(x)=\sin2x(1+\cos2x)^{-1}$ and $g(x)=\frac{1+\sin x-\cos x}{1+\sin x+\cos x}$.

**9.** Find the area of the triangle formed with vertices $(0,0)$, $\left(\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right],0\right)$ and $\left(0,\lim\limits_{x\to0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}\right)$, where $[\cdot]$ denotes the greatest integer function.

**10.** Prove that the straight lines whose direction cosines are given by the relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$.

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

## SOLUTIONS

**1. (b) :** Let $\theta = \frac{\pi}{16} \Rightarrow 8\theta = \frac{\pi}{2}$

I^st and last gives $\tan^2\theta + \cot^2\theta = \frac{8}{1-\cos 4\theta} - 2$

Similarly other terms.

**2. (c) :** $f(x) = e^{\sin(x-[x])\cos \pi x}$

$\sin(x - [x]) = \sin\{x\}$. Period is 1

$\cos\pi x$, Period is 2

Hence $f(x)$ is of period 2

**3. (d) :** $f(x, y)$ is homogeneous function of degree $n \in R$ in $x, y$ if $f(kx, ky) = k^n f(x, y)$ ; where $k > 0$

**4. (c) : (A)** $\lim_{x\to 0} \ln\left(1-2\sin^2\frac{x}{2}\right)^{1/x}$

$$= \lim_{x\to 0} \ln\left[\left(1-2\sin^2\frac{x}{2}\right)^{\frac{-1}{2\sin^2\frac{x}{2}}}\right]^{\frac{-2\sin^2\frac{x}{2}}{x}} = \lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{x} = 0$$

**(B)** $\lim_{x\to 1}\frac{1-x}{\ln x} = \lim_{h\to 0}\frac{h}{\ln(1-h)} = \lim_{h\to 0}\frac{1}{\ln[(1-h)^{-1/h}]^{-1}} = -1$

**(C)** $\lim_{x\to 0}\frac{\frac{x-\sin x}{x^3}}{\frac{x-\tan x}{x^3}} = \frac{\frac{1}{6}}{-\frac{1}{3}} = -\frac{1}{2}$

**5. (c)**

**6.** We have $\cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}}$ ... (1)

$$= \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}} = \sqrt{4-3\sec^2 x}$$

$\Rightarrow \cos^2 y = 4 - 3\sec^2 x = 4 - 3(1 + \tan^2 x)$

$= 1 - 3\tan^2 x$

$\Rightarrow \sin^2 y = 3\tan^2 x \Rightarrow \sin y = \sqrt{3}\tan x$

$\Rightarrow \cos y\frac{dy}{dx} = \sqrt{3}\sec^2 x$

$\therefore \frac{dy}{dx} = \sqrt{\frac{3}{\cos x\cdot\cos 3x}}$ [from (1)]

**7.** $\because \phi : R \to R$ is continuous at $x = a$ and satisfies $f(x) - f(a) = \phi(x)(x - a)\ \forall\ x \in R$

$\Rightarrow \frac{f(x)-f(a)}{x-a} = \phi(x)$

$\Rightarrow \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\phi(x)$

$\Rightarrow f'(a) = \phi(a)$ $[\because \lim_{x\to a}\phi(x) = \phi(a)]$

$\Rightarrow f$ is differentiable at $x = a$.

**8.** Given $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + \beta + \gamma) = 0$

$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + \beta + \gamma) = 0$

where $\beta, \gamma \in (0, \pi)$

$\Rightarrow [\cos\alpha + \cos(\alpha + \beta)]^2 + [\sin\alpha + \sin(\alpha + \beta)]^2 = 1$

$\Rightarrow 2 + 2[\cos\beta] = 1$

$\therefore \cos\beta = -\frac{1}{2}$. Similarly, $\cos\gamma = -\frac{1}{2}$,

$\therefore \beta = \gamma = \frac{2\pi}{3}$

Now, $f(x) = \frac{\sin 2x}{1+\cos 2x} = \tan x$ and $g(x) = \tan\frac{x}{2}$

$\therefore f'\left(\frac{2\pi}{3}\right) = \sec^2\frac{2\pi}{3} = 4$ and $\lim_{x\to\frac{2\pi}{3}} g(x) = \tan\frac{\pi}{3} = \sqrt{3}$

**9.** Let $O = (0, 0)$

$$A = \left(\lim_{x\to\frac{\pi}{2}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right], 0\right) = (-2, 0)$$

$$B = \left(0, \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}\right) = (0, 1)$$

$\therefore$ Area of $\Delta OAB = \frac{1}{2}|-2-0| = 1$ sq. unit

**10.** We have $al + bm + cn = 0$ ... (1)

$fmn + gnl + hlm = 0$ ... (2)

Eliminate $n$, we get

$\Rightarrow ag\left(\frac{l}{m}\right)^2 + (af + bg + ch)\left(\frac{l}{m}\right) + bf = 0$ ... (3)

Now, if $l_1, m_1, n_1$ and $l_2, m_2, n_2$ are d.c.'s of two lines then roots of (3) are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

$\therefore$ product of the roots $= \frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{bf}{ag}$

$\therefore \frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b}$

$\therefore \frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$

$\because$ lines are perpendicular

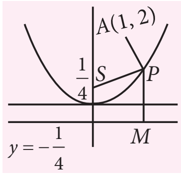
$\therefore l_1l_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

# MATHS MUSING

## SOLUTION SET-165

**1.** **(c) :** Focus is $\left(0, \frac{1}{4}\right)$




Directrix is $y = -\frac{1}{4}$

$PS + PA = PM + PA$, takes minimum value when $A$, $P$, $M$ are collinear.

$\therefore$ Minimum value $= 2 + \frac{1}{4} = \frac{9}{4}$

**2.** **(a) :** The number of all 5 digit numbers is $6! - 5! = 600$

Numbers divisible by 11 without digit 1 are

23045 with 8 permutations

32450 with 8 permutations

Likewise there are 16 numbers without 3 and 16 numbers without 5

$\therefore$ The number of numbers divisible by 11 are $16 \times 3 = 48$

Probability $= \frac{48}{600} = \frac{2}{25}$

**3.** **(b):** $\sum_{r=1}^{10} \frac{1}{(2r-1)(2r+1)(2r+3)}$

$$= \frac{1}{8}\left(1 - \frac{2}{3} + \frac{1}{3} - \frac{1}{21} + \frac{1}{23}\right) = \frac{40}{483} = \frac{m}{n}$$

$\therefore \quad m + n = 523$

**4.** **(d):** Let $\alpha, \beta$ be the roots. $\alpha + \beta = -a$, $\alpha\beta = 6a$

Eliminating $a$, we get $(\alpha + 6)(\beta + 6) = 36$

$\therefore \quad \alpha = -6 + d, \ \beta = -6 + \frac{36}{d}$

The number of pairs $(\alpha, \beta)$, is the number of divisors of 36.

*i.e.*, $d = \pm 1, \ \pm 2, \pm 3, \pm 4, \pm 6$

$\therefore$ The number of values of $a$ is 10.

**5.** **(c) :** Let $a = \cos\theta$, $b = \sin\theta$

$$z = \cos\theta + i\sin\theta = \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^2$$

$$= \frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}} = \frac{c+i}{c-i}, \text{ where,}$$

$$c = \cot\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta} = \frac{1+a}{b}$$

**6.** **(a, b, c, d) :** $\frac{\sin^3\theta}{\cos^3\theta} + \frac{\cos^3\theta}{\sin^3\theta} = 12 + \frac{8}{\sin^3 2\theta}$

$\Rightarrow \quad \sin^6\theta + \cos^6\theta = \frac{3}{2}\sin^3 2\theta + 1$

$\Rightarrow \quad 1 - \frac{3}{4}\sin^2 2\theta = \frac{3}{2}\sin^3 2\theta + 1$

$\Rightarrow \quad \sin^2 2\theta \ (2\sin 2\theta + 1) = 0$

$\Rightarrow \quad \sin 2\theta = -\frac{1}{2} = \sin\left(\frac{7\pi}{6}\right)$

$\Rightarrow \quad 2\theta = n\pi + (-1)^n\left(\frac{7\pi}{6}\right)$

$\Rightarrow \quad \theta = \frac{n\pi}{2} + (-1)^n\frac{7\pi}{12}; \theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

**7.** **(c) :** $\frac{dx}{dy} - x = y \Rightarrow xe^{-y} = \int ye^{-y}\,dy = -(y+1)e^{-y} + A$

$x = 0, y = 0 \Rightarrow A = 1$ and $x = e^y - (y+1)$

At $y = \ln 3$, $\frac{dx}{dy} = 3 - 1 = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{2}$

**8.** **(d):** $\int_0^1 x\,dy = \int_0^1 (e^y - y - 1)\,dy = e - \frac{5}{2}$

**9.** **(6):** The minimum number 1 occurs as $2^{nd}$, $3^{rd}$, ... $9^{th}$ term in the sequence.

$$\therefore N = \sum_{r=2}^{9}\binom{9}{r-1} = \binom{9}{1} + \binom{9}{2} + \ldots + \binom{9}{8} = 2^9 - 2 = 510$$

**10.** **(P) $\to$ 4; (Q) $\to$ 1; (R) $\to$ 3; (S) $\to$ 2**

(P) The desired number is the coefficient of $x^{10}$ in $(x + x^2 + \ldots + x^6)^4$ or coefficient of $x^6$ in $(1 + x + \ldots + x^5)^4$

$= (1 - x^6)^4 \ (1 - x)^{-4}$

$$= (1 - 4x^6 + \ldots)\left(1 + \binom{4}{1}x + \binom{5}{2}x^2 + \ldots\right)$$

Required coefficient $= \binom{9}{6} - 4 = 84 - 4 = 80$

(Q) $a = 2\alpha - 1$, $b = 2\beta - 1$, $c = 2\gamma - 1$, $d = 2\delta - 1$

$\Rightarrow \quad \alpha + \beta + \gamma + \delta = 10$

The number of solutions is $\binom{9}{3} = 84$

(R) $(1 + (x+2y))^5 = 1 + \binom{5}{1}(x+2y) + \binom{5}{2}(x+2y)^2 + \binom{5}{3}(x+2y)^3 + \ldots.$

The coefficient of $x^2y = \binom{5}{3}\cdot 3 \cdot 2 = 60$

(S) $\sum_{r=1}^{10}\frac{r\cdot C_r}{C_{r-1}} = \sum_{r=1}^{10}\frac{r(10-r+1)}{r} = \sum_{r=1}^{10}(11-r)$

$$= 10 + 9 + 8 + \ldots + 1 = \frac{10\cdot 11}{2} = 55$$

# YOU ASK WE ANSWER

**Do you have a question that you just can't get answered?**

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

**1.** If $A$, $B$, $C$ be the centres of three co-axial circles and $t_1$, $t_2$, $t_3$ be the lengths of the tangents to them from any point, then prove that

$$\overline{BC}\cdot t_1^2+\overline{CA}\cdot t_2^2+\overline{AB}t_3^2=0$$

***Kishor, U.P.***

**Ans.** Let the equations of three circles are

$x^2+y^2+2g_ix+c=0$, $i=1, 2, 3$.

According to the question

$A\equiv(-g_1, 0)$, $B\equiv(-g_2, 0)$, $C\equiv(-g_3, 0)$

Let any point be $P(h, k)$

$$t_1=\sqrt{h^2+k^2+2g_1h+c}$$

$$t_2=\sqrt{h^2+k^2+2g_2h+c}$$

$$t_3=\sqrt{h^2+k^2+2g_3h+c}$$

and $\overline{AB}=(g_1-g_2)$

$\overline{BC}=(g_2-g_3)$

and $\overline{CA}=(g_3-g_1)$

Now $\overline{BC}\cdot t_1^2+\overline{CA}\cdot t_2^2+\overline{AB}\cdot t_3^2$

$=\Sigma(g_2-g_3)\,(h^2+k^2+2g_1h+c)$

$=(h^2+k^2+c)\,\Sigma\,(g_2-g_3)+2h\Sigma g_1(g_2-g_3)$

$=(h^2+k^2+c)\,(g_2-g_3+g_3-g_1+g_1-g_2)$

$\quad+2h\,\{g_1(g_2-g_3)+g_2(g_3-g_1)+g_3(g_1-g_2)\}.$

$=(h^2+k^2+c)\,(0)+2h(0)$

$=0$

which proves the result.

**2.** If $f(x)=|x+1|\,\{|x|+|x-1|\}$, then draw the graph of $f(x)$ in the interval $[-2, 2]$ and discuss the continuity and differentiability in $[-2, 2]$.

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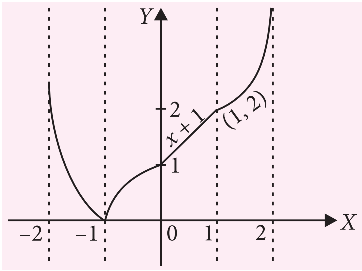
**Ans.** Here, $f(x)=|x+1|\,\{|x|+|x-1|\}$

$$f(x)=\begin{cases}(x+1)(2x-1)\,; & -2\le x<-1\\ -(x+1)(2x-1); & -1\le x<0\\ (x+1); & 0\le x<1\\ (x+1)\,(2x-1)\,; & 1\le x\le 2\end{cases}$$

Thus the graph of $f(x)$ is;




which is clearly, continuous for $x\in[-2, 2]$ and differentiable for $x\in[-2, 2]-\{-1, 0, 1\}$

**3.** If $n$ is a positive integer, show that

$$\frac{n!}{(x+1)(x+2)....(x+n)}=\frac{C_1}{x+1}-\frac{(2!)C_2}{x+2}+\frac{(3!)C_3}{x+3}-....+(-1)^{n+1}\frac{(n!)C_n}{x+n}$$

***Mehul, Assam***

**Ans.** If the L.H.S of the given expression is decomposed into partial fractions, we have

$$\frac{n!}{(x+1)(x+2)\cdots(x+n)}=\frac{A_1}{x+1}+\frac{A_2}{x+2}+\cdots+\frac{A_n}{x+n}$$

*i.e.*, $n!=A_1(x+2)\,(x+3)\cdots(x+n)+A_2(x+1)$ $(x+3)\cdots(x+n)+\cdots+A_n(x+1)\,(x+2)$ $\cdots(x+n-1)$ ...(i)

Putting $x=-1$ in (i), we have

$n!=A_1\cdot 1\cdot 2\cdot 3\cdots\cdots(n-1)$

*i.e.*, $A_1=\dfrac{n!}{1\cdot 2\cdot 3\cdots\cdots(n-1)}=n={}^nC_1$

Similarly, putting $x=-2, -3, \cdots\cdots, -n$ in (i) respectively, we have

$$A_2=\frac{n!}{-1\cdot 2\cdots\cdots(n-2)}=-n(n-1)=-{}^nC_2(2!)$$

$$A_3=\frac{n!}{1\cdot 2\cdots\cdots(n-3)}=n(n-1)(n-2)={}^nC_3(3!)$$

$\vdots$

and $A_n=(-1)^{n+1}\,{}^nC_n\,(n!)$

Hence, we have

$$\frac{n!}{(x+1)(x+2)\cdots(x+n)}=\frac{C_1}{x+1}-\frac{(2!)C_2}{x+2}+\frac{(3!)C_3}{x+3}-....+(-1)^{n+1}\frac{(n!)C_n}{x+n}$$

which is the desired result.